U(N) tools for Loop Quantum Gravity: The Return of the Spinor

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We explore the classical setting for the U(N) framework for SU(2) intertwiners for loop quantum gravity (LQG) and describe the corresponding phase space in terms of spinors with the appropriate constraints. We show how its quantization leads back to the standard Hilbert space of intertwiner states defined as holomorphic functionals. We then explain how to glue these intertwiners states in order to construct spin network states as wave-functions on the spinor phase space. In particular, we translate the usual loop gravity holonomy observables to our classical framework. Finally, we propose how to derive our phase space structure from an action principle which induces non-trivial dynamics for the spin network states. We conclude by applying explicitly our framework to states living on the simple 2-vertex graph and discuss the properties of the resulting Hamiltonian.

Contents

Introduction

Loop quantum gravity (LQG) is now a well-established approach to quantum gravity. It proposes a canonical framework with quantum states -the spin network states- defining the 3d space geometry and whose evolution in time generates space-time. The main challenges still faced by the theory are, on the one hand, getting a full understanding of the geometric meaning of the spin network states both at the Planck scale and in a semi-classical regime and, on the

other hand, constructing a consistent dynamics which would lead back to the standard dynamics of the gravitational field at large scales. These two issues are obviously related and can not truly be solved independently.

The present work builds on the previously introduced U(N) framework for intertwiners in loop quantum gravity [1–4]. Intertwiners are the building blocks of the spin network states, which are constructed from gluing intertwiners together along particular graphs. This framework was shown to be particularly efficient in building coherent semi-classical intertwiner states [3, 4], which could be a useful basis to define full coherent semi-classical spin network states. A side-product of this approach is the possibility of reformulating the whole LQG spin network framework in terms of spinors [3]. From the point of view of the U(N) techniques, this comes from the harmonic oscillators that are used to build all the operators and Hilbert spaces and which can be understood as the quantization of spinors. From the point of view of loop quantum gravity, re-writing everything in terms of spinors might look like a return to the origin since the theory was first developed in spinorial notations. We nevertheless believe that this spinorial reformulation is relevant to understand better the geometric meaning of the spin network states and should be useful in studying their semi-classical behavior and writing the quantum gravity dynamics.

This perspective is consistent with the recent "twisted geometry" framework developed by one of the authors and collaborators [5, 6]. They explain how the classical phase space of loop quantum gravity on a fixed graph can be expressed in terms of spinors and show how this can be used to explore the relation between spin network states and discrete (Regge) geometries. This is particularly relevant to understanding the physical meaning of spinfoam models defining the dynamics for spin networks.

In the present paper, we show how to fully recast the U(N) framework for SU(2) intertwiners in terms of spinors. More precisely, we define the corresponding classical spinor phase space and introduce a classical action principle from which we derive that phase space structure. Furthermore we show how its quantization leads to the Hilbert space of intertwiner states. These intertwiners are built as some particular holomorphic functionals of the spinors. We then move on to full spin network states. We explain how to glue intertwiners together to build spin networks. This leads us to define the classical spinor phase space behind the Hilbert space of spin network states built on a fixed graph. In particular, we explain how to translate the usual LQG holonomy observables in our framework. Then, similarly to the case of a single intertwiner, we describe the corresponding classical action principle and discuss the possible interaction terms we can add to the action in order to define a non-trivial dynamics for the spin network states of quantum geometry. Finally, we apply these techniques to spin networks on the 2-vertex graph and compare the resulting classical action principle to the 2-vertex quantum gravity model previously constructed by some of the authors [7].

Spinors and Notations

In this preliminary section, we introduce spinors and the related useful notations, following the previous works [3, 4, 6]. Considering a spinor z,

$$|z\rangle = \begin{pmatrix} z^0 \\ z^1 \end{pmatrix}, \qquad \langle z| = \begin{pmatrix} \bar{z}^0 & \bar{z}^1 \end{pmatrix},$$

we associate to it a geometrical 3-vector $\vec{V}(z)$, defined from the projection of the 2×2 matrix $|z\rangle\langle z|$ onto Pauli matrices σ_a (taken Hermitian and normalized so that $(\sigma_a)^2 = \mathbb{I}$):

$$|z\rangle\langle z| = \frac{1}{2}\left(\langle z|z\rangle \mathbb{I} + \vec{V}(z)\cdot\vec{\sigma}\right). \tag{1}$$

The norm of this vector is obviously $|\vec{V}(z)| = \langle z|z\rangle = |z^0|^2 + |z^1|^2$ and its components are given explicitly as:

$$V^z = |z^0|^2 - |z^1|^2, \qquad V^x = 2\,\Re\,(\bar{z}^0 z^1), \qquad V^y = 2\,\Im\,(\bar{z}^0 z^1). \tag{2}$$

The spinor z is entirely determined by the corresponding 3-vector $\vec{V}(z)$ up to a global phase. We can give the reverse map:

$$z^0 = e^{i\phi}\sqrt{\frac{|\vec{V}| + V^z}{2}}, \quad z^1 = e^{i(\phi-\theta)}\sqrt{\frac{|\vec{V}| - V^z}{2}}, \quad \tan\theta = \frac{V^y}{V^x}, \tag{3}$$

where $e^{i\phi}$ is an arbitrary phase.

Following [3], we also introduce the map duality ς acting on spinors:

$$\varsigma \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} = \begin{pmatrix} -\bar{z}^1 \\ \bar{z}^0 \end{pmatrix}, \qquad \varsigma^2 = -1. \tag{4}$$

This is an anti-unitary map, $\langle \varsigma z|\varsigma w\rangle = \langle w|z\rangle = \overline{\langle z|w\rangle}$, and we will write the related state as

$$|z] \equiv \varsigma|z\rangle, \qquad [z|w] = \overline{\langle z|w\rangle}.$$

This map ς maps the 3-vector $\vec{V}(z)$ onto its opposite:

$$|z][z| = \frac{1}{2}\left(\langle z|z\rangle \mathbb{I} - \vec{V}(z)\cdot\vec{\sigma}\right). \tag{5}$$

Finally considering the setting necessary to describe intertwiners with N legs, we consider N spinors z_i and their corresponding 3-vectors $\vec{V}(z_i)$. Typically, we can require that the N spinors satisfy a closure condition, i.e that the sum of the corresponding 3-vectors vanishes, $\sum_i \vec{V}(z_i) = 0$. Coming back to the definition of the 3-vectors $\vec{V}(z_i)$, the closure condition is easily translated in terms of 2×2 matrices:

$$\sum_i |z_i\rangle\langle z_i| = A(z)\mathbb{I}, \qquad \text{with} \quad A(z) \equiv \frac{1}{2}\sum_i \langle z_i|z_i\rangle = \frac{1}{2}\sum_i |\vec{V}(z_i)|. \tag{6}$$

This further translates into quadratic constraints on the spinors:

$$\sum_i z_i^0\,\bar{z}_i^1 = 0, \quad \sum_i \left|z_i^0\right|^2 = \sum_i \left|z_i^1\right|^2 = A(z). \tag{7}$$

In simple terms, it means that the two components of the spinors, z_i^0 and z_i^1, are orthogonal N-vectors of equal norm.

I. OVERVIEW OF THE U(N) STRUCTURE OF INTERWINERS

Here, we quickly review the U(N) formalism for SU(2) intertwiners in loop quantum gravity. This framework was introduced and improved in a series of papers [1–4, 7]. More precisely, intertwiners with N legs are SU(2)-invariant states in the tensor product of N (irreducible) representations of SU(2). Then the basic tool used to define the U(N) formalism is the Schwinger representation of the $\mathfrak{su}(2)$ Lie algebra in terms of a pair of harmonic oscillators. Since we would like to describe the tensor product of N SU(2)-representations, we will need N copies of the $\mathfrak{su}(2)$-algebra and thus we consider N pairs of harmonic oscillators a_i, b_i with i running from 1 to N.

The local $\mathfrak{su}(2)$ generators acting on each leg i are defined as quadratic operators:

$$J_i^z = \frac{1}{2}(a_i^\dagger a_i - b_i^\dagger b_i), \qquad J_i^+ = a_i^\dagger b_i, \qquad J_i^- = a_i b_i^\dagger, \qquad E_i = (a_i^\dagger a_i + b_i^\dagger b_i). \tag{8}$$

The J_i's satisfy the standard commutation algebra while the total energy E_i is a Casimir operator:

$$[J_i^z, J_i^\pm] = \pm J_i^\pm, \qquad [J_i^+, J_i^-] = 2J_i^z, \qquad [E_i, \vec{J}_i] = 0. \tag{9}$$

The operator E_i is the total energy carried by the pair of oscillators a_i, b_i and simply gives twice the spin $2j_i$ of the corresponding SU(2)-representation. Indeed, we can easily express the standard SU(2) Casimir operator in terms of this energy:

$$\vec{J}_i^2 = \frac{E_i}{2}\left(\frac{E_i}{2}+1\right) = \frac{E_i}{4}\left(E_i+2\right). \tag{10}$$

In the context of loop quantum gravity, the spin j_i given as the value of the operator $E_i/2$ defines the area associated to the leg i of the intertwiner.

Then we look for operators invariant under global SU(2) transformations generated by $\vec{J} \equiv \sum_i \vec{J}_i$. The key result, which is the starting point of the U(N) formalism, is that we can identify quadratic invariant operators acting on pairs of (possibly equal) legs i, j [1, 3]:

$$E_{ij} = a_i^\dagger a_j + b_i^\dagger b_j, \qquad E_{ij}^\dagger = E_{ji}, \tag{11}$$

$$F_{ij} = (a_i b_j - a_j b_i), \qquad F_{ji} = -F_{ij}. \tag{12}$$

These operators $E, F, F^\dagger$ form a closed algebra:

$$\begin{aligned}
[E_{ij}, E_{kl}] &= \delta_{jk} E_{il} - \delta_{il} E_{kj}, \\
[E_{ij}, F_{kl}] &= \delta_{il} F_{jk} - \delta_{ik} F_{jl}, \qquad [E_{ij}, F^\dagger_{kl}] = \delta_{jk} F^\dagger_{il} - \delta_{jl} F^\dagger_{ik}, \\
[F_{ij}, F^\dagger_{kl}] &= \delta_{ik} E_{lj} - \delta_{il} E_{kj} - \delta_{jk} E_{li} + \delta_{jl} E_{ki} + 2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \\
[F_{ij}, F_{kl}] &= 0, \qquad [F^\dagger_{ij}, F^\dagger_{kl}] = 0.
\end{aligned} \tag{13}$$

First focusing on the E_{ij} operators, their commutators close and they form a $\mathfrak{u}(N)$-algebra. This is why this formalism has been dubbed the $\mathrm{U}(N)$ framework for loop quantum gravity [1, 2]. The diagonal operators are equal to the previous operators giving the energy on each leg, $E_{ii} = E_i$. Then the value of the total energy $E \equiv \sum_i E_i$ gives twice the sum of all spins $2 \times \sum_i j_i$, i.e twice the total area in the context of loop quantum gravity.

The E_{ij}-operators change the energy/area carried by each leg, while still conserving the total energy, while the operators F_{ij} (resp. $F^\dagger_{ij}$) will decrease (resp. increase) the total area E by 2:

$$[E, E_{ij}] = 0, \qquad [E, F_{ij}] = -2F_{ij}, \quad [E, F^\dagger_{ij}] = +2F^\dagger_{ij}. \tag{14}$$

This suggests to decompose the Hilbert space of N-valent intertwiners into subspaces of constant area:

$$\mathcal{H}_N = \bigoplus_{\{j_i\}} \mathrm{Inv}\left[\otimes_{i=1}^N V^{j_i}\right] = \bigoplus_{J\in\mathbb{N}} \bigoplus_{\sum_i j_i = J} \mathrm{Inv}\left[\otimes_{i=1}^N V^{j_i}\right] = \bigoplus_J \mathcal{H}_N^{(J)}, \tag{15}$$

where V^{j_i} denote the Hilbert space of the irreducible SU(2)-representation of spin j_i, spanned by the states of the oscillators a_i, b_i with fixed total energy $E_i = 2j_i$.

It was proven in [2] that each subspace $\mathcal{H}_N^{(J)}$ of N-valent intertwiners with fixed total area J carries an irreducible representation of $\mathrm{U}(N)$ generated by the E_{ij} operators. These are representations with Young tableaux given by two horizontal lines with equal numbers of cases (J). More constructively, we can characterize them by their highest weight vector $v_N^{(J)}$:

$$E_1\,|v_N^{(J)}\rangle = J\,|v_N^{(J)}\rangle, \quad E_2\,|v_N^{(J)}\rangle = J\,|v_N^{(J)}\rangle, \quad E_{k\geq 3}\,|v_N^{(J)}\rangle = 0, \quad E\,|v_N^{(J)}\rangle = 2J\,|v_N^{(J)}\rangle, \qquad E_{i<j}\,|v_N^{(J)}\rangle = 0\,. \tag{16}$$

These highest weight vectors define bivalent intertwiners where all the area is carried by two legs $i = 1, 2$ while all the other legs carried the trivial SU(2)-representation $j_{k\geq 3} = 0$. Then the operators E_{ij} allow to navigate from state to state within each subspace $\mathcal{H}_N^{(J)}$. On the other hand, the operators F_{ij}, $F^\dagger_{ij}$ allow to go from one subspace $\mathcal{H}_N^{(J)}$ to the next $\mathcal{H}_N^{(J\pm 1)}$, thus endowing the full space of N-valent intertwiners with a Fock space structure with creation operators $F^\dagger_{ij}$ and annihilation operators F_{ij}.

Finally, the identification of the highest vectors was made possible by realizing that the operators E_{ij} satisfy quadratic constraints, which can then be turned by conditions relating the quadratic $\mathrm{U}(N)$-Casimir operator to the total area E [2]. Then it was realized that the whole set of operators $E_{ij}, F_{ij}, F^\dagger_{ij}$ satisfy quadratic constraints [7]:

$$\forall i, j, \quad \sum_k E_{ik} E_{kj} = E_{ij} \left(\frac{E}{2} + N - 2\right), \tag{17}$$

$$\sum_k F^\dagger_{ik} E_{jk} = F^\dagger_{ij}\,\frac{E}{2}, \qquad \sum_k E_{jk} F^\dagger_{ik} = F^\dagger_{ij} \left(\frac{E}{2} + N - 1\right), \tag{18}$$

$$\sum_k E_{kj} F_{ik} = F_{ij} \left(\frac{E}{2} - 1\right), \qquad \sum_k F_{ik} E_{kj} = F_{ij} \left(\frac{E}{2} + N - 2\right), \tag{19}$$

$$\sum_k F^\dagger_{ik} F_{kj} = E_{ij} \left(\frac{E}{2} + 1\right), \qquad \sum_k F_{kj} F^\dagger_{ik} = (E_{ij} + 2\delta_{ij}) \left(\frac{E}{2} + N - 1\right). \tag{20}$$

As already noticed in [7], these relations look a lot like constraints on the multiplication of two matrices E_{ij} and F_{ij}. This is the point which we will explore further in the present paper, and we will show that the operators E_{ij} and F_{ij} are truly the quantization of the matrix elements of two $N \times N$ classical matrices built from a set of $2N$ spinors. This will allow to explore further the representation of the intertwiner space $\mathcal{H}_N$ as the L^2 over the Grassmanian space $U(N)/(\mathrm{U}(2) \times \mathrm{U}(N-2))$ introduced in [2, 3].

II. THE CLASSICAL SETTING FOR INTERTWINERS

A. The Matrix Algebra

Drawing inspiration from the operators E_{ij} and F_{ij} and the quadratic constraints relating them, our goal is to describe the classical system behind the Hilbert space of SU(2)-intertwiners. Thus we consider two $N \times N$ matrices, a Hermitian matrix and an antisymmetric one:

$$M = M^\dagger, \quad {}^tQ = -Q. \tag{21}$$

We assume that they satisfy the same constraints (17-20) that the operators E_{ij} and F_{ij}, up to terms which we identify as coming from quantum ordering :

$$\begin{aligned} M^2 &= \frac{\mathrm{Tr}M}{2}\,M, \qquad & Q\bar{Q} &= -\frac{\mathrm{Tr}M}{2}\,M, \\ MQ &= \frac{\mathrm{Tr}M}{2}\,Q, \qquad & \bar{Q}M &= \frac{\mathrm{Tr}M}{2}\,\bar{Q}, \end{aligned} \tag{22}$$

where Q actually plays the role of $F^\dagger$ while $\bar{Q}$ corresponds to F.

Let us now solve these equations and parameterize the space of solutions to these constraints.

Result 1. *The quadratic constraints on M and Q, together with the requirement of Hermiticity of M and anti-symmetry of Q, entirely fix these two matrices up to a global* U(N) *transformation and a relative phase:*

$$\begin{aligned} M &= \lambda\, U\Delta U^{-1}, \qquad & \Delta &= \left(\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & 0_{N-2} \end{array}\right), \\ Q &= e^{i\theta}\lambda\, U\Delta_\epsilon {}^tU, \qquad & \Delta_\epsilon &= \left(\begin{array}{cc|c} & 1 & \\ -1 & & \\ \hline & & 0_{N-2} \end{array}\right), \end{aligned} \tag{23}$$

where U is a unitary matrix $U^\dagger U = \mathbb{I}$, $\lambda \in \mathbb{R}$ and $\exp(i\theta)$ is an arbitrary phase.

Proof. Let us start with the trivial case when $\mathrm{Tr}M = 0$. Then it is obvious to see that $M = Q = 0$. Let us thus assume that $\mathrm{Tr}M \neq 0$ and let us define $\lambda = (\mathrm{Tr}M)/2$.

The equation $M^2 = \lambda M$ implies that the matrix M is a projector, with two eigenvalues 0 and λ. Then using that $\lambda = (\mathrm{Tr}M)/2$, we can conclude that its rank is two. Thus we can write $M = \lambda U \Delta U^{-1}$, in terms of a unitary matrix $U \in \mathrm{U}(N)$ defining an orthonormal basis diagonalizing M. More precisely, calling $|e_i\rangle$ the canonical basis for N-vectors, the basis $U\,|e_i\rangle$ diagonalizes M:

$$M\,U|e_1\rangle = \lambda\,U|e_1\rangle, \quad M\,U|e_2\rangle = \lambda\,U|e_2\rangle, \quad M\,U|e_{k\geq 3}\rangle = 0.$$

The next step is to determine the matrix Q in terms of λ and U. We first apply the condition that $\bar{Q}M = \lambda\,\bar{Q}$, this implies that:

$$\bar{Q}\,U|e_{k\geq 3}\rangle = 0,$$

which is equivalent to $Q\,\bar{U}|e_{k\geq 3}\rangle = 0$. Then we can use the condition $MQ = \lambda Q$ to determine the action of Q on the first two basis vectors:

$$\forall i = 1, 2, \quad M\,Q\,\bar{U}|e_i\rangle = \lambda\,Q\,\bar{U}|e_i\rangle.$$

Looking at the state $\bar{U}|e_1\rangle$, this means that either $Q\,\bar{U}|e_1\rangle = 0$ or that $Q\,\bar{U}|e_1\rangle$ is in the subspace generated by $U|e_1\rangle$ and $U|e_2\rangle$. The first possibility is impossible, since it would imply that $MU|e_1\rangle = 0$ due to the condition $Q\bar{Q} = -\lambda M$. Thus we write:

$$Q\,\bar{U}|e_1\rangle = \alpha U|e_1\rangle + \beta U|e_2\rangle.$$

Moreover, since Q is antisymmetric, we have $\langle Ue_1|Q|\bar{U}e_1\rangle = 0$ and thus the coefficient α vanishes. Further using the antisymmetry property, we have $\langle Ue_1|Q|\bar{U}e_2\rangle = -\langle Ue_2|Q|\bar{U}e_1\rangle$, thus we get:

$$Q\,\bar{U}|e_1\rangle = \beta U|e_2\rangle, \qquad Q\,\bar{U}|e_2\rangle = -\beta U|e_1\rangle.$$

Finally, using the last condition $Q\bar{Q} = -\lambda M$, we can compute the value of the coefficient β:

$$|\beta|^2 \, U|e_1\rangle = -Q\bar{Q}\, U|e_1\rangle = \lambda M\, U|e_1\rangle = \lambda^2\, U|e_1\rangle \qquad \Rightarrow \qquad \beta = e^{i\theta}\lambda,$$

where θ is an arbitrary angle. This allows to conclude that: $Q = e^{i\theta}\lambda\, U\Delta_\epsilon {}^t U$ since ${}^tU = \bar{U}^{-1}$.

Reversely, it is easy to check that the resulting expressions for M and Q in terms of U, λ, θ always satisfy the quadratic constraints which we started from.

□

From now on, using the $U(1)$ freedom of chosing U, we will set the phase θ to 0 and define the two matrices as:

$$M = \lambda\, U\Delta U^{-1}, \qquad Q = \lambda\, U\Delta_\epsilon {}^t U. \tag{24}$$

Comparing with the U(N) framework for intertwiners reviewed in the previous section, the rank-2 matrix Δ plays the role of the highest weight vector, from which we will get the full space of states by acting on it with U(N) transformations. Looking at the stabilizer group for the diagonal matrix Δ, we see that M is invariant under:

$$U \to UV, \quad \forall V \in \mathrm{U}(2) \times \mathrm{U}(N-2), \tag{25}$$

and therefore the matrix M exactly parameterizes the coset space $\mathrm{U}(N)/\mathrm{U}(2)\times\mathrm{U}(N-2)$, which was already identified in [2, 3] as the classical space behind N-valent SU(2) intertwiners. Similarly looking at the stabilizer group for Δ_ϵ, we realize that Q is invariant under a slightly smaller subgroup:

$$U \to UV, \quad \forall V \in \mathrm{SU}(2) \times \mathrm{U}(N-2). \tag{26}$$

Therefore, the space of functions $f(Q)$ invariant under multiplication by a phase, $f(Q) = f(e^{i\theta}Q)$, is isomorphic to the space of functions on the Grassmannian space $\mathrm{U}(N)/\mathrm{U}(2) \times \mathrm{U}(N-2)$. This is consistent with the fact that the quadratic conditions on M and Q only determine the matrix Q up to a phase.

Finally we will also require the positivity of the matrix M, i.e $\lambda \geq 0$. This reflects the positivity of the energy/area E at the quantum level. So that we now work with $\lambda \in \mathbb{R}^+$.

B. From U(N) Matrices to Spinors and the Closure Condition

We start by writing explicitly the two matrices M and Q in terms of the matrix elements of the unitary transformation U:

$$M_{ij} = \lambda(u_{i1}\bar{u}_{j1} + u_{i2}\bar{u}_{j2}), \qquad Q_{ij} = \lambda(u_{i1}u_{j2} - u_{i2}u_{j1}). \tag{27}$$

These expressions only involve the first columns of the matrix U. This comes from the definition of the diagonal matrices Δ and Δ_ϵ, and the resulting invariance under the $\mathrm{U}(N-2)$ subgroup. Comparing these equations with the definitions (11-12) of the operators E_{ij} and $F^\dagger_{ij}$, we see that the matrix element u_{i1} corresponds to the operator $a^\dagger_i$. Following this logic of a classical-quantum correspondence, we define the spinors z_i as the rescaled first two columns of the U-matrix:

$$z_i \equiv \begin{pmatrix} \bar{u}_{i1}\sqrt{\lambda} \\ \bar{u}_{i2}\sqrt{\lambda} \end{pmatrix}. \tag{28}$$

The matrices M and Q are easily expressed in terms of these spinors:

$$M_{ij} = \langle z_i | z_j \rangle = \overline{\langle z_j | z_i \rangle}, \qquad Q_{ij} = \langle z_j | z_i] = \overline{[z_i | z_j\rangle} = -\overline{[z_j | z_i\rangle}. \tag{29}$$

The matrix elements Q_{ij} are clearly anti-holomorphic in the z_i's while the M_{ij}'s mix both holomorphic and anti-holomorphic components. With M and Q written as such, the quadratic constraints on M and Q are exactly the relations between the matrices $\langle z_i|z_j\rangle$ and $[z_j|z_i\rangle$ written in [3]. The spinors z_i are not entirely free, since they come from the unitary matrix U. The only constraint is that the two vectors u_{i1} and u_{i2} are part of an orthonormal matrix, that is that they are of unit-norm and orthogonal:

$$U^\dagger U = \mathbb{I} \qquad \Rightarrow \qquad \sum_i |u_{i1}|^2 = \sum_i |u_{i2}|^2 = 1, \quad \sum_i u_{i1}\bar{u}_{i2} = 0. \tag{30}$$

This is easy translated into conditions on the spinors:

$$\sum_i |z_i^0|^2 = \sum_i |z_i^1|^2 = \lambda, \quad \sum_i z_i^0 \bar{z}_i^1 = 0. \tag{31}$$

Checking out the short preliminary section about spinors, we see that these conditions correspond exactly to the *closure constraints* on the spinors z_i, thus they are equivalent to the following conditions:

$$\sum_i \vec{V}(z_i) = 0, \quad \sum_i |z_i\rangle\langle z_i| = \lambda \mathbb{I}, \qquad \frac{1}{2}\sum_i |\vec{V}(z_i)| = \frac{1}{2}\sum_i \langle z_i|z_i\rangle = \lambda. \tag{32}$$

Thus the requirement of unitarity, that our matrix U lays in $\mathrm{U}(N)$, is equivalent to the closure conditions on our N spinors. We could relax these closure conditions by dropping the requirement of unitarity, but this would break the quadratic constraints that M and Q satisfy.

Let us introduce the matrix elements of the 2×2 matrix $\sum_i |z_i\rangle\langle z_i|$:

$$\mathcal{C}_{ab} = \sum_i z_i^a \bar{z}_i^b. \tag{33}$$

Then the unitary or closure conditions are written very simply:

$$\mathcal{C}_{00} - \mathcal{C}_{11} = 0, \quad \mathcal{C}_{01} = \mathcal{C}_{10} = 0. \tag{34}$$

C. Phase Space and $\mathrm{SU}(2)$ Invariance

Let us introduce a simple Poisson bracket on our space of N spinors:

$$\{z_i^a, \bar{z}_j^b\} \equiv i\,\delta^{ab}\delta_{ij}, \tag{35}$$

with all other brackets vanishing, $\{z_i^a, z_j^b\} = \{\bar{z}_i^a, \bar{z}_j^b\} = 0$. This is exactly the Poisson bracket for $2N$ decoupled harmonic oscillators.

We start by checking that the closure conditions generates global $\mathrm{SU}(2)$ transformations on the N spinors. First, we can compute the Poisson brackets between the various components of the $\mathcal{C}$-constraints :

$$\begin{aligned} &\{\mathcal{C}_{00} - \mathcal{C}_{11}, \mathcal{C}_{01}\} = -2i\mathcal{C}_{01}, \quad \{\mathcal{C}_{00} - \mathcal{C}_{11}, \mathcal{C}_{10}\} = +2i\mathcal{C}_{10}, \quad \{\mathcal{C}_{10}, \mathcal{C}_{01}\} = i(\mathcal{C}_{00} - \mathcal{C}_{11}), \\ &\{\mathrm{Tr}\mathcal{C}, \mathcal{C}_{00} - \mathcal{C}_{11}\} = \{\mathrm{Tr}\mathcal{C}, \mathcal{C}_{01}\} = \{\mathrm{Tr}\mathcal{C}, \mathcal{C}_{10}\} = 0. \end{aligned} \tag{36}$$

These four components $\mathcal{C}_{ab}$ do indeed form a closed $\mathfrak{u}(2)$ algebra with the three closure conditions $\mathcal{C}_{00} - \mathcal{C}_{11}$, $\mathcal{C}_{01}$ and $\mathcal{C}_{10}$ forming the $\mathfrak{s}u(2)$ subalgebra. Thus we will write $\vec{\mathcal{C}}$ for these three $\mathfrak{s}u(2)$-generators with $\mathcal{C}^z \equiv \mathcal{C}_{00} - \mathcal{C}_{11}$ and $\mathcal{C}^+ = \mathcal{C}_{10}$ and $\mathcal{C}^- = \mathcal{C}_{01}$. We can further check their commutator with the spinors themselves:

$$\begin{array}{lll} \{\mathcal{C}_{00} - \mathcal{C}_{11}, z_i^0\} = -i\,z_i^0, & \{\mathcal{C}_{01}, z_i^0\} = 0, & \{\mathcal{C}_{10}, z_i^0\} = -iz_i^1, \\ \{\mathcal{C}_{00} - \mathcal{C}_{11}, z_i^1\} = +i\,z_i^1, & \{\mathcal{C}_{01}, z_i^1\} = -iz_i^0, & \{\mathcal{C}_{10}, z_i^1\} = 0, \end{array} \tag{37}$$

which indeed generates the standard $\mathrm{SU}(2)$ transformations on the N spinors. The three closure conditions $\vec{\mathcal{C}}$ will actually become the generators $\vec{J}$ at the quantum level, while the operator $\mathrm{Tr}\,\mathcal{C}$ will correspond to the total energy/area E.

We also compute the Poisson brackets of the M_{ij} and Q_{ij} matrix elements:

$$\begin{aligned} &\{M_{ij}, M_{kl}\} = i(\delta_{kj}M_{il} - \delta_{il}M_{kj}), \\ &\{M_{ij}, Q_{kl}\} = i(\delta_{jk}Q_{il} - \delta_{jl}Q_{ik}), \\ &\{Q_{ij}, Q_{kl}\} = 0, \\ &\{\bar{Q}_{ij}, Q_{kl}\} = i(\delta_{ik}M_{lj} + \delta_{jl}M_{ki} - \delta_{jk}M_{li} - \delta_{il}M_{kj}), \end{aligned} \tag{38}$$

which reproduces the expected commutators (13) up to the i-factor. We further check that these variables commute with the closure constraints generating the $\mathrm{SU}(2)$ transformations:

$$\{\vec{\mathcal{C}}, M_{ij}\} = \{\vec{\mathcal{C}}, Q_{ij}\} = 0. \tag{39}$$

Finally, we look at their commutator with $\mathrm{Tr}\,\mathcal{C}$. On the one hand, we have:

$$\{\mathrm{Tr}\,\mathcal{C}, M_{ij}\} = 0, \tag{40}$$

which confirms that the matrix M is invariant under the full U(2) subgroup. On the other hand, we compute:

$$\{\mathrm{Tr}\,\mathcal{C}, Q_{ij}\} = \{\sum_k M_{kk}, Q_{ij}\} = +2i\, Q_{ij}, \tag{41}$$

which means that $\mathrm{Tr}\,\mathcal{C}$ acts as a dilatation operator on the Q variables, or reversely that the Q_{ij} acts as creation operators for the total energy/area variable $\mathrm{Tr}\,\mathcal{C}$.

D. Action and Matrix model

We can derive the previous Poisson bracket from an action principle, which directly defines the classical phase space associated to SU(2) intertwiners. In terms of the spinor variables, the action simply reads:

$$\begin{aligned} S_0[z_i] \;&\equiv\; \int dt\, \left(\sum_{i,a} i z_i^a \partial_t \bar{z}_i^a - \Lambda^{ab}\mathcal{C}_{ab}\right) \;=\; \int dt\, \left(\sum_{i,a} i z_i^a \partial_t \bar{z}_i^a - \Lambda^{ab}\sum_i z_i^a \bar{z}_i^b\right), \\ &=\; \int dt\, \left(\sum_i -i\langle z_i|\partial_t z_i\rangle + \langle z_i|\Lambda|z_i\rangle\right), \end{aligned} \tag{42}$$

where the 2×2 matrix elements Λ_{ab} are Lagragian multipliers satisfying $\mathrm{Tr}\,\Lambda = \sum_a \Lambda_{aa} = 0$ and enforcing the closure constraints $\vec{\mathcal{C}} = 0$. As we have seen in the previous sections, the three constraints $\vec{\mathcal{C}}$ are first class constraints, they generate global SU(2) transformations on the spinors. We must both impose and solve these closure constraints and identify SU(2)-invariant observables on the space of constrained spinors.

This is the free action defining only the classical kinematics on the intertwiner space described in terms of spinors. We can define dynamics by adding an interaction term to the action:

$$S[z_i] \;\equiv\; S_0[z_i] + I[z_i] \;=\; \int dt\, \left(\sum_{i,a} i z_i^a \partial_t \bar{z}_i^a - \Theta^{ab}\mathcal{C}_{ab} - H[z_i]\right),$$

where $H[z_i]$ would be the Hamiltonian of the system defining how intertwiners evolve.

We can re-write this action principle in terms of the initial unitary matrix U and the parameter $\lambda \in \mathbb{R}^+$:

$$\begin{aligned} S_0[U,\lambda] \;&\equiv\; \int dt\, \left[-i\,\mathrm{Tr}\,\sqrt{\lambda}U\Delta\partial_t(\sqrt{\lambda}U^\dagger) \;-\; \mathrm{Tr}\Theta\,(UU^\dagger - \mathbb{I})\right] \\ &=\; \int dt\, \left(-i\,\lambda \mathrm{Tr}\,U\Delta\partial_t U^\dagger \;-i\,\partial_t\lambda \;-\; \mathrm{Tr}\Theta\,(UU^\dagger - \mathbb{I})\right), \end{aligned} \tag{43}$$

where the $N\times N$ matrix Θ is a Lagrange multiplier enforcing that the matrix U is unitary.

A first remark is that the kinematical term $\sqrt{\lambda}\partial_t\sqrt{\lambda}$ is a total derivative and does not induce any evolution, thus the dynamics of the variable λ is entirely determined by its coupling to U through the kinematical term $\lambda\mathrm{Tr}\,U\Delta\partial_t U^\dagger$. Therefore, if the unitary matrix U does not evolve, then λ is frozen too. Then, we see that the action is invariant under the left U(N) action:

$$U \to VU, \quad V \in \mathrm{U}(N),$$

for a constant unitary matrix V (independent from t), but is only invariant under the right action of the stabilizer subgroup $\mathrm{U}(2)\times\mathrm{U}(N-2)$:

$$U \to UV, \quad V \in \mathrm{U}(2)\times\mathrm{U}(N-2).$$

If we want to add dynamics to this system, it is natural to require that the interaction term be also invariant under the same symmetries. This greatly constrains the possible terms of a Hamiltonian. Indeed, we are left with

polynomials of $\mathrm{T}r(\lambda U\Delta U^{-1})^k = \mathrm{T}r\, M^k = 2\lambda^k$, which are simply polynomials in λ. Since the variable λ is not really dynamical, we conclude that there are no truly non-trivial invariant dynamics for a single intertwiner.

We could bypass this conclusion by allowing Hamiltonian operators that break the $\mathrm{U}(N)$ symmetry. We do not really see the purpose of such procedure, although it could be used to model the coupling of a single intertwiner to an external source breaking the $\mathrm{U}(N)$ invariance. On the other hand, we will see in section.IV that we can have non-trivial dynamics as soon as we work with many intertwiners when considering true spin network states on an arbitrary graph.

Finally, we end this section stressing that we have managed to reformulate the classical setting of a single SU(2) intertwiner as a unitary matrix model, which was the original goal of the paper [1] which introduces the $\mathrm{U}(N)$ framework for SU(2) intertwiners.

E. Intertwiners as (Anti-)Holomorphic Functionals

Now that we have fully characterize the phase space associated to the spinors z_i and the variables M_{ij}, Q_{ij}, we can proceed to the quantization.

The most natural choice is to consider polynomials in the Q_{ij} matrix elements. More precisely, we introduce the Hilbert spaces $\mathcal{H}_J^{(Q)}$ of homogeneous polynomials in the Q_{ij} of degree J:

$$\mathcal{H}_J^{(Q)} \equiv \{P \in \mathbb{P}[Q_{ij}] \,|\quad P(\rho Q_{ij}) = \rho^J\, P(Q_{ij}),\, \forall \rho \in \mathbb{C}\}\,. \tag{44}$$

These are polynomials completely anti-holomorphic in the spinors z_i (or holomorphic in $\bar{z}_i$) and of order $2J$. Let us point out that the variables Q_{ij} are not independent, since they are expressed in terms of the spinors z_i. Resultingly, they are related to each other by the Plücker relations as already noticed in [3]:

$$Q_{ij} = \bar{z}_i^0 \bar{z}_j^1 - \bar{z}_i^1 \bar{z}_j^0 \qquad \Rightarrow \quad Q_{ij}Q_{kl} = Q_{il}Q_{kj} + Q_{ik}Q_{jl}. \tag{45}$$

Interestingly, this can be interpreted as the recoupling relation between interchanging the four legs (i,j,k,l) of the intertwiner (see fig.1).

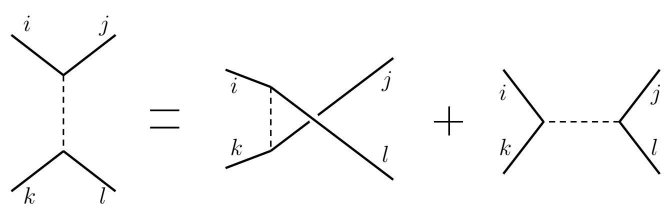

FIG. 1: Focusing on the four legs (i,j,k,l) of the intertwiner, the Plücker relation $Q_{ij}Q_{kl} = Q_{il}Q_{kj} + Q_{ik}Q_{jl}$ on the Q-variables becomes the (standard) recoupling relation for SU(2) intertwiners (more precisely, for holonomy operators acting on SU(2) intertwiners). This relation is often used in Loop Quantum Gravity when still using states defined as products of Wilson loops instead of spin network states.

Our claim is that these Hilbert spaces $\mathcal{H}_J^{(Q)}$ are isomorphic to the Hilbert space $\mathcal{H}_N^{(J)}$ of N-valent intertwiners with fixed total area J. To this purpose, we will construct the explicit representation of the operator quantizing M_{ij} and Q_{ij} on the spaces $\mathcal{H}_J^{(Q)}$ and show that they match the actions of the $\mathrm{U}(N)$ operators E_{ij} and $F_{ij}^\dagger$ which we described earlier. Our quantization relies on quantizing the $\bar{z}_i$ as multiplication operators while promoting z_i to a derivative operator:

$$\widehat{\bar{z}}_i^a \equiv \bar{z}_i^a \times\,, \qquad \widehat{z}_i^a \equiv \frac{\partial}{\partial \bar{z}_i^a}, \tag{46}$$

which satisfies the commutator $[\hat{z}, \hat{\bar{z}}] = 1$ as expected for the quantization of the classical bracket $\{z, \bar{z}\} = i$. Then,

we quantize the matrix elements M_{ij} and Q_{ij} and the closure constraints following this correspondence:

$$\widehat{M}_{ij} \;=\; \bar{z}_i^0 \frac{\partial}{\partial \bar{z}_j^0} + \bar{z}_i^1 \frac{\partial}{\partial \bar{z}_j^1}\,, \tag{47}$$

$$\widehat{Q}_{ij} \;=\; \bar{z}_i^0 \bar{z}_j^1 - \bar{z}_i^1 \bar{z}_j^0 \;=\; Q_{ij}\,, \tag{48}$$

$$\widehat{\bar{Q}}_{ij} \;=\; \frac{\partial^2}{\partial \bar{z}_i^0 \partial \bar{z}_j^1} - \frac{\partial^2}{\partial \bar{z}_i^1 \partial \bar{z}_j^0}\,, \tag{49}$$

$$\widehat{\mathcal{C}}_{ab} \;=\; \sum_k \bar{z}_k^b \frac{\partial}{\partial \bar{z}_k^a}. \tag{50}$$

It is straightforward to check that the $\widehat{\mathcal{C}}_{ab}$ and the $\widehat{M}_{ij}$ respectively form a $\mathfrak{u}(2)$ and a $\mathfrak{u}(N)$ Lie algebra, as expected:

$$[\widehat{\mathcal{C}}_{ab}, \widehat{\mathcal{C}}_{cd}] \;=\; \delta_{ad}\widehat{\mathcal{C}}_{cb} - \delta_{cb}\widehat{\mathcal{C}}_{ad}, \qquad [\widehat{M}_{ij}, \widehat{M}_{kl}] \;=\; \delta_{kj}\widehat{M}_{il} - \delta_{il}\widehat{M}_{kj}, \qquad [\widehat{\mathcal{C}}_{ab}, \widehat{M}_{ij}] \;=\; 0. \tag{51}$$

which amounts to multiply the Poisson bracket (36) and (38) by $-i$. Then, we first check the action of the closure constraints on functions of the variables Q_{ij} :

$$\widehat{\vec{\mathcal{C}}}\, Q_{ij} \;=\; 0, \qquad \widehat{(\mathrm{Tr}\,\mathcal{C})}\, Q_{ij} \;=\; 2Q_{ij},$$

$$\forall P \in \mathcal{H}_J^{(Q)} = \mathbb{P}_J[Q_{ij}], \qquad \widehat{\vec{\mathcal{C}}}\, P(Q_{ij}) \;=\; 0, \qquad \widehat{(\mathrm{Tr}\,\mathcal{C})}\, P(Q_{ij}) \;=\; 2J\, P(Q_{ij}), \tag{52}$$

so that our wavefunctions $P \in \mathcal{H}_J^{(Q)}$ are SU(2)-invariant (vanish under the closure constraints) and are eigenvectors of the $\mathrm{Tr}\,\mathcal{C}$-operator with eigenvalue $2J$.

Second, we check that the operators $\widehat{M}$ and $\widehat{(\mathrm{Tr}\,\mathcal{C})}$ satisfy the same quadratic constraints on the Hilbert space $\mathcal{H}_J^{(Q)}$ (i.e assuming that the operators acts on SU(2)-invariant functions vanishing under the closure constraints) that the $\mathfrak{u}(N)$-generators E_{ij}:

$$\widehat{(\mathrm{Tr}\,\mathcal{C})} = \sum_k \widehat{M}_{kk}, \qquad \sum_k \widehat{M}_{ik}\widehat{M}_{kj} \;=\; \widehat{M}_{ij}\left(\frac{\widehat{(\mathrm{Tr}\,\mathcal{C})}}{2} + N - 2\right), \tag{53}$$

which allows us to get the value of the (quadratic) $U(N)$-Casimir operator on the space $\mathcal{H}_J^{(Q)}$:

$$\sum_{ik} \widehat{M}_{ik}\widehat{M}_{ki} \;=\; \widehat{(\mathrm{Tr}\,\mathcal{C})}\left(\frac{\widehat{(\mathrm{Tr}\,\mathcal{C})}}{2} + N - 2\right) \;=\; 2J(J+N-2).$$

Thus, we can safely conclude that this provides a proper quantization of our spinors and M-variables, which matches exactly with the $\mathfrak{u}(N)$-structure on the intertwiner space (with the exact same ordering):

$$\mathcal{H}_J^{(Q)} \sim \mathcal{H}_N^{(J)}, \quad \widehat{M}_{ij} \;=\; E_{ij}, \quad \widehat{(\mathrm{Tr}\,\mathcal{C})} \;=\; E. \tag{54}$$

Now, turning to the $\widehat{Q}_{ij}$-operators, it is straightforward to check that they have the exact same action that the $F_{ij}^\dagger$ operators, they satisfy the same Lie algebra commutators (13) and the same quadratic constraints (18-20). Clearly, the simple multiplicative action of an operator $\widehat{Q}_{ij}$ send a polynomial in $\mathbb{P}_J[Q_{ij}]$ to a polynomial in $\mathbb{P}_{J+1}[Q_{ij}]$. Reciprocally, the derivative action of $\widehat{\bar{Q}}_{ij}$ decreases the degree of the polynomials and maps $\mathbb{P}_{J+1}[Q_{ij}]$ onto $\mathbb{P}_J[Q_{ij}]$.

Finally, let us look at the scalar product on whole space of polynomials $\mathbb{P}[Q_{ij}]$. In order to ensure the correct Hermicity relations for $\widehat{M}_{ij}$ and $\widehat{Q}_{ij}, \widehat{\bar{Q}}_{ij}$, it seems that we have a unique[1] measure (up to a global factor):

$$\forall \phi, \psi \in \mathbb{P}[Q_{ij}], \quad \langle \phi | \psi \rangle \;\equiv\; \int \prod_i d^4 z_i\, e^{-\sum_i \langle z_i | z_i \rangle}\, \overline{\phi(Q_{ij})}\, \psi(Q_{ij})\,. \tag{55}$$

[1] If we ask to recover only the hermiticity relation for $\widehat{M}_{ij}$ then we can use any function of $\sum_i \langle z_i | z_i \rangle$ as a measure instead of the exponential. Asking in turns that $\widehat{Q}$ and $\widehat{\bar{Q}}$ are hermitian conjugate fixes entirely the measure up to a scale.

Then it is easy to check that we have $\widehat{M}^\dagger_{ij} = \widehat{M}_{ji}$ and $\widehat{Q}^\dagger_{ij} = \widehat{\bar{Q}}_{ij}$ as wanted.

It is easy to see that the spaces of homogeneous polynomials $\mathbb{P}_J[Q_{ij}]$ are orthogonal with respect to this scalar product. The quickest way to realize that this is true is to consider the operator $\widehat{(\mathrm{Tr}\,\mathcal{C})}$, which is Hermitian with respect to this scalar product and takes different values on the spaces $\mathbb{P}_J[Q_{ij}]$ depending on the value of J. Thus these spaces $\mathbb{P}_J[Q_{ij}]$ are orthogonal to each other[2].

This concludes our quantization procedure thus showing that the intertwiner space for N legs and fixed total area $J = \sum_i j_i$ can be seen as the space of homogeneous polynomials in the Q_{ij} variables with degree J. This provides us with a description of the intertwiners as wave-functions anti-holomorphic in the spinors z_i (or equivalently holomorphic in $\bar{z}_i$) constrained by the closure conditions. In particular, the highest weight vector of the $\mathrm{U}(N)$ representation $\mathbb{P}_J[Q_{ij}]$ is the monomial Q_{12}^J, which defines the (unique) bivalent intertwiner carrying the spin $\frac{J}{2}$ on both legs 1 and 2. Finally, in this context, the Plücker relation on the Q_{ij} variables can truly be interpreted as recoupling relations on intertwiners.

Before moving on, we would like to comment about the equivalence on using the spinor variables or the Q_{ij} variables or the initial λ, U variables. Indeed, at the end of the day, we are considering (anti-holomorphic) functions of the spinors z_i satisfying the closure conditions $\mathcal{C}$ and also invariant under the $\mathrm{SU}(2)$ transformations that they generate, this defines the manifold $\mathbb{C}^{4N}//\mathrm{SU}(2) = \mathbb{C}^{4N}/\mathrm{SL}(2,\mathbb{C})$, with dimension:

$$4N - (3+3).$$

Let us now compare with the Q-matrix defined as $Q = \lambda U \Delta_\epsilon {}^tU$. As we already said earlier, this defines the manifold $\mathbb{R}_+ \times \mathrm{U}(N)/(\mathrm{SU}(2) \times \mathrm{U}(N-2))$ since the expression of Q is invariant under $U \to UV$ with $V \in \mathrm{SU}(2) \times \mathrm{U}(N-2)$. It is easy to compute the dimension of this manifold:

$$1 + N^2 - 3 - (N-2)^2 \,=\, 4N - 6,$$

which coincides exactly (as expected!) with the previous dimension of the spinor manifold.

In the next section, we will present an alternative construction, which can be considered as "dual" to the representation defined above. It is based on the coherent states for the oscillators, thus recovering the framework of the $\mathrm{U}(N)$ coherent intertwiner states introduced in [3] and further developed in [4].

F. Intertwiners as Holomorphic Functionals - version 2

We can also build our Hilbert space of quantum states as a Fock space acting with creation operators on the vacuum state $|0\rangle$ of the oscillators. This will be based on the $\mathrm{SU}(2)$ coherent intertwiners and $\mathrm{U}(N)$ coherent states as defined in [3, 4].

We start by quantizing the spinors components $z_i^0, \bar{z}_i^0$ and $z_i^1, \bar{z}_i^1$ satisfying the classical Poisson bracket (35) as the creation and annihilation operators of harmonic oscillators, respectively $a_i, a_i^\dagger$ and $b_i, b_i^\dagger$. We will have these operators acting on the standard coherent states for quantum oscillators by multiplication by z and derivative with respect to z.

More precisely, let us begin by introducing the basis of $\mathrm{SU}(2)$ coherent intertwiners as defined in [3, 4] in terms of the spinors $z_i \in \mathbb{C}^2$ and some extra spin labels $j_i \in \mathbb{N}/2$:

$$||\{j_i, z_i\}\rangle \,\equiv\, \int_{\mathrm{SU}(2)} dg\, g \triangleright \prod_i \frac{(z_i^0 a_i^\dagger + z_i^1 b_i^\dagger)^{2j_i}}{\sqrt{(2j_i)!}}\,|0\rangle, \tag{57}$$

where $|0\rangle$ is the vacuum states of the harmonic oscillators, $a_i\,|0\rangle \,=\, b_i\,|0\rangle \,=0$. The group-averaging is taken over $\mathrm{SU}(2)$ with its standard action on spinors as 2×2 matrices. This is exactly the $\mathrm{SU}(2)$ transformations generated by

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[2] If $\phi_J(Q_{ij})\,\psi_J(Q_{ij})$ are homogeneous of degree J we can express this scalar product as an integral over the grassmanian:

$$\langle \phi_J | \psi_J \rangle \,=\, (N+J-1)!(N+J-2)! \int_{G_{2,N}} \overline{\phi_J(Q_{ij})}\,\psi_J(Q_{ij})\,, \tag{56}$$

where $G_{2,N} = \{|z_i\rangle_{i=1\cdots n} | \sum_i |z_i\rangle\langle z_i| = 1\}$.

$\vec{J}$ [3, 4]. We further introduce the U(N) coherent states[3]:

$$|J,\{z_i\}\rangle \equiv \sum_{\sum_i j_i=J} \frac{1}{\sqrt{(2j_i)!}}\,||\{j_i,z_i\}\rangle = \frac{1}{(2J)!}\int_{\mathrm{SU}(2)} dg\, g \triangleright \prod_i \left(\sum_i (z_i^0 a_i^\dagger + z_i^1 b_i^\dagger)\right)^{2J} |0\rangle. \tag{58}$$

Now, we can define our operators $\widehat{M}_{ij}$, $\widehat{Q}_{ij}$ and $\widehat{\bar{Q}}_{ij}$ as differential operators in the z_k's acting in the basis $|J,\{z_i\}\rangle$ and we can check that they exactly match the action of the operators E_{ij},F_{ij} and $F^\dagger_{ij}$.

Result 2. *We define the operators $\widehat{M}_{ij}$, $\widehat{Q}_{ij}$ and $\widehat{\bar{Q}}_{ij}$ as differential operators acting on holomorphic functionals $|\varphi\rangle \equiv \int [d^2z]^{2N}\,\varphi(z_k)\,|J,\{z_k\}\rangle$:*

$$\begin{aligned}
\widehat{M}_{ij} &= -\left(\frac{\partial}{\partial z_i^0} z_j^0 + \frac{\partial}{\partial z_i^1} z_j^1\right) = -\left(z_j^0\frac{\partial}{\partial z_i^0} + z_j^1\frac{\partial}{\partial z_i^1}\right) - 2\delta_{ij} \\
\widehat{Q}_{ij} &= \frac{\partial^2}{\partial z_i^0 \partial z_j^1} - \frac{\partial^2}{\partial z_i^1 \partial z_j^0}. \\
\widehat{\bar{Q}}_{ij} &= z_i^0 z_j^1 - z_i^1 z_j^0 = \bar{Q}_{ij}
\end{aligned} \tag{59}$$

These differential operators exactly reproduce the action of respectively the operators $E_{ij} = a_i^\dagger a_j + b_i^\dagger b_j$, $F_{ij} = a_i b_j - a_j b_i$ and $F^\dagger_{ij}$ on the (coherent) states $|J,\{z_k\}\rangle$.

Proof. We start by computing the action of the operators E_{ij},F_{ij} and $F^\dagger_{ij}$ on the states $||\{j_k,z_k\}\rangle$. In order to do this, we use the definition of those states and simply compute the commutator of the E_{ij},F_{ij} and $F^\dagger_{ij}$ with the operators $(z_k^0 a_k^\dagger + z_k^1 b_k^\dagger)^{2j_k}$. Then it is straightforward to get:

$$\begin{aligned}
E_{ij}\,||\{j_k,z_k\}\rangle &= \frac{\sqrt{2j_j}}{\sqrt{2j_i+1}}\left(z_j^0\frac{\partial}{\partial z_i^0} + z_j^1\frac{\partial}{\partial z_i^1}\right)||\{j_i+\frac{1}{2}, j_j-\frac{1}{2}, j_k, z_k\}\rangle, \\
F_{ij}\,||\{j_k,z_k\}\rangle &= \sqrt{2j_j}\sqrt{2j_i}\,(z_i^0 z_j^1 - z_i^1 z_j^0)\,||\{j_i-\frac{1}{2}, j_j-\frac{1}{2}, j_k, z_k\}\rangle, \\
F^\dagger_{ij}\,||\{j_k,z_k\}\rangle &= \frac{1}{\sqrt{(2j_i+1)}\sqrt{2j_i+1}}\left(\frac{\partial^2}{\partial z_i^0 \partial z_j^1} - \frac{\partial^2}{\partial z_i^1 \partial z_j^0}\right)||\{j_i+\frac{1}{2}, j_j+\frac{1}{2}, j_k, z_k\}\rangle,
\end{aligned} \tag{60}$$

which allows to obtain the action on the $|J,\{z_k\}\rangle$ states:

$$\begin{aligned}
E_{ij}\,|J,\{z_k\}\rangle &= \left(z_j^0\frac{\partial}{\partial z_i^0} + z_j^1\frac{\partial}{\partial z_i^1}\right)|J,\{z_k\}\rangle, \\
F_{ij}\,|J,\{z_k\}\rangle &= (z_i^0 z_j^1 - z_i^1 z_j^0)\,|J-1,\{z_k\}\rangle, \\
F^\dagger_{ij}\,|J,\{z_k\}\rangle &= \left(\frac{\partial^2}{\partial z_i^0 \partial z_j^1} - \frac{\partial^2}{\partial z_i^1 \partial z_j^0}\right)|J+1,\{z_k\}\rangle,
\end{aligned} \tag{61}$$

These expressions were actually already derived [4] by other means. Using these actions of the operators E_{ij},F_{ij} and $F^\dagger_{ij}$ on the states $||J,\{z_k\}\rangle$, we finally derive their action on states $\int [d^2z]^{2N}\,\varphi(z_k)\,|J,\{z_k\}\rangle$ by integration by parts and we recover the expressions given above. □

As in the previous section, we can check that these operators $\widehat{M}_{ij}$, $\widehat{Q}_{ij}$ and $\widehat{\bar{Q}}_{ij}$ satisfy the exact expected commutation relations and quadratic constraints (when acting on SU(2)-invariant states), and thus provide a proper

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[3] As was shown in [4], these states are closely related to the coherent state basis for the quantum oscillators:

$$\sum_{J\in\mathbb{N}} \beta^{2J}\,|J,\{z_i\}\rangle = \int dg\, g \triangleright e^{\beta[\sum_i z_i^0 a_i^\dagger + z_i^1 b_i^\dagger]}\,|0\rangle.$$

This works because the integral over SU(2) of odd powers of $a^\dagger$ and $b^\dagger$ vanishes.

quantization of our classical Poisson structure (38). We notice that it is the operator $\widehat{\overline{Q}}_{ij} = F_{ij}$ which now acts as a multiplication operator while $\widehat{Q}_{ij} = F^{\dagger}_{ij}$ becomes a derivative operator. In this sense, we can consider this quantization scheme as "dual" to the one presented in the previous section. This comes from quantizing z as $\partial_{\bar{z}}$ in the previous scheme while quantizing $\bar{z}$ as $-\partial_z$ in the present scheme based on the coherent states. For more details on the U(N) coherent state basis, the interested reader can refer to [3, 4].

III. BUILDING HOLONOMIES FOR LOOP GRAVITY

Up to now, we have described the Hilbert space of a single intertwiner, corresponding to a single vertex of a spin network state, in terms of spinors and U(N) operators. More precisely, we have described the classical system of $2N$ spinors constrained by the closure conditions, which is isomorphic to the coset space U(N)/SU(2) $\times$ U($N-2$), and we have explained how its quantization leads back to the space of N-valent intertwiner states.

In this section, we discuss the generalization of this framework to whole spin network states for Loop Quantum Gravity. We explain how to glue intertwiners, or more precisely how to glue these systems of spinors together along particular graphs. The main result is how to express holonomies in terms of the spinors. This allows to view spin network states as functionals of our Q_{ij} variables and fully reformulate the kinematics of Loop Quantum Gravity in terms of spinors and the U(N) operators.

A. Revisiting Spin Network States

Building on the previous works on the U(N) framework for intertwiners [2, 3, 7] and the twistor representation of twisted geometries for loop gravity [5, 6], we would like to give a full representation of the spin network states in terms of spinors.

Let us start by considering a given oriented graph Γ, with E edges and V vertices. Let us call $s(e)$ and $t(e)$ respectively the source and target vertices of each edge. Then the Hilbert space of cylindrical functions for Loop Quantum Gravity consists in all functions of E group elements $g_e \in \mathrm{SU}(2)$ which are invariant under the SU(2)-action at each vertex:

$$\forall h_v \in \mathrm{SU}(2)^{\times V}, \quad \phi(g_e) = \phi(h_{s(e)} g_e h^{-1}_{t(e)}). \tag{62}$$

The scalar product between two such functionals is defined by the straightforward integration with respect to the Haar measure on SU(2):

$$\langle \phi | \widetilde{\phi} \rangle \,=\, \int_{\mathrm{SU}(2)^{\times E}} [dg_e]\, \overline{\phi(g_e)}\, \widetilde{\phi}(g_e) \tag{63}$$

so that the Hilbert space of SU(2)-invariant cylindrical functions on the considered graph Γ is

$$\mathcal{H}_\Gamma \,\equiv\, L^2(\mathrm{SU}(2)^E/\mathrm{SU}(2)^V).$$

A basis of this space is given by the spin network states, which are labeled by one SU(2)-representation j_e on each edge e and one intertwiner state I_v on each vertex v of the graph. One goal is to make the link between this and our formalism based on spinors, Q_{ij} variables and U(N) operators.

The first step was already described in [2]. We consider one intertwiner state constructed with the U(N) formalism, and then we glue them along the edges of the graph. More precisely, we start with a function $\psi(Q^1, .., Q^V)$ where Q^v is the $N_v \times N_v$ matrix corresponding to the vertex v where N_v is the valence of the node v. Each of these matrices Q^v is constructed from a set of spinors $z_{v,e}$ attached to the corresponding vertex v. These intertwiners are decoupled for now. Following [2], we glue them by requiring that they carry the same spin j_e from the point of view of both vertices $s(e)$ and $t(e)$. Since the spin on the leg e of an intertwiner at the vertex v is given by the energy operator $E^v_e = \widehat{M}^v_{ee}$ living on that leg, this amounts to imposing the constraint $E^{s(e)}_e - E^{t(e)}_e = 0$ on each edge e. This *matching condition* corresponds to the classical constraint:

$$M^{s(e)}_{ee} - M^{T(e)}_{ee} = \langle z_{s(e),e} | z_{s(e),e} \rangle - \langle z_{t(e),e} | z_{t(e),e} \rangle = 0, \tag{64}$$

which requires that the two spinors $z_{s(e),e}$ and $z_{t(e),e}$ have equal norm. At the quantum level, this constraint imposes a U(1)-invariance for each edge:

$$\psi(z_{s(e),e}, z_{t(e),e}) \,=\, \psi(e^{i\theta_e}\, z_{s(e),e}, e^{-i\theta_e} z_{t(e),e}), \qquad \forall \theta_e \in [0, \pi]\,, \tag{65}$$

$$\psi(Q^{s(e)},Q^{t(e)},Q^v) \,=\, \psi(e^{-i(\delta_{ie}+\delta_{je})\theta_e}\,Q^{s(e)}_{ij},e^{+i(\delta_{ie}+\delta_{je})\theta_e}\,Q^{t(e)}_{ij},Q^v), \qquad \forall\theta_e\in[0,\pi]\,, \tag{66}$$

whether we express the wave-functions in terms of the Q^v_{ij} matrix elements or directly in terms of the spinors $z_{v,e}$. Notice that we multiply the source and target spinors by opposite phases.

There is two equivalent ways to impose these matching constraints on the wave-functions:

- Either, we impose $\langle z_{s(e),e}|z_{s(e),e}\rangle - \langle z_{t(e),e}|z_{t(e),e}\rangle = 0$ at the classical level on the phase space and consider equivalence classes of spinors under the corresponding $\mathrm{U}(1)^E$ transformations; and then quantize the system by considering (anti-)holomorphic wave-functions on this constrained phase space.
- Or quantize the system of intertwiners as we have done up to now without imposing $\langle z_{s(e),e}|z_{s(e),e}\rangle - \langle z_{t(e),e}|z_{t(e),e}\rangle = 0$ at the classical level, and then impose the $\mathrm{U}(1)^E$ invariance to the resulting (anti-)holomorphic wave-functions.

Conjecture 1. *Following this procedure, we consider (anti-)holomorphic wave-functions of the spinors $\psi(z_{s(e),e},z_{t(e),e})$, where all sets of spinors around each vertex v satisfy the closure conditions and invariant under* $\mathrm{SU}(2)$ *(generated by those same closure conditions), and such that they are invariant under multiplication by a phase on each edge e. We conjecture that the L^2 space of such functions with respect to the measure (55) is isomorphic to the Hilbert space $\mathcal{H}_\Gamma$ of spin network states of the graph Γ. In more mathematical terms:*

$$\begin{aligned}\mathcal{H}_\Gamma = L^2(\mathrm{SU}(2)^E/\mathrm{SU}(2)^V) \,&=\, L^2_{holo}\left(\times_v\,\mathbb{C}^{2N_v}//\mathrm{SU}(2)\right)\,/\mathrm{U}(1)^E \\ &=\, L^2_{holo}\big(\times_v\,\mathbb{R}^+\times\mathrm{U}(N_v)/(\mathrm{SU}(2)\times\mathrm{U}(N_v-2))\big)\,/\mathrm{U}(1)^E\end{aligned} \tag{67}$$

where the $//\mathrm{SU}(2)$ quotient means that we both impose the closure conditions and the invariance under the $\mathrm{SU}(2)$ *transformations that they generate. In this scheme, it is truly the closure conditions at each vertex that induce the* $\mathrm{SU}(2)$*-gauge invariance of our quantum states.*

A first hint towards establishing this conjecture is a count of the degrees of freedom. Starting by focusing on a given vertex v, we are looking at holomorphic functions of N_v spinors satisfying the closure conditions $\mathcal{C}$ and invariant under the $\mathrm{SU}(2)$, which gives:

$$\frac{1}{2}\,[4N_v-(3+3)]\,=2N_v-3,$$

taking into account that each spinor counts for 4 real degrees of freedom and the $\frac{1}{2}$-factor accounts for considering only holomorphic functions. We have already commented in section.II E on the equivalence of counting the number of degrees of freedom defined by the spinor variables or by the Q variables. We now sum over all vertices v and impose the $\mathrm{U}(1)$ on each edge, which gives:

$$\sum_v(2N_v-3)-E\,=\,3E-3V,$$

since the combinatorics of a graph ensures that the number of edges can be expressed in terms of the valence of all the nodes as $2E=\sum_v N_v$. We compare this to the dimension of the quotient manifold $\mathrm{SU}(2)^E/SU(2)^V$ whose dimension is obviously $3(E-V)$ since $\mathrm{SU}(2)$ has dimension 3 (excluding the "degenerate" case when $E=V$ which corresponds to a single Wilson loop).

The second step towards establishing the correspondence between the standard formalism of loop (quantum) gravity and our spinor formulation is provided by the reconstruction of the group element g_e in terms of the spinors. This was done in [6].

Considering an edge e with the two spinors at each of its end-vertices $z_{s(e),e}$ and $z_{t(e),e}$, there exists a unique $\mathrm{SU}(2)$ group element mapping one onto the other. More precisely:

$$g_e\,\equiv\,\frac{|z_{s(e),e}]\langle z_{t(e),e}|-|z_{s(e),e}\rangle[z_{t(e),e}|}{\sqrt{\langle z_{s(e),e}|z_{s(e),e}\rangle\langle z_{t(e),e}|z_{t(e),e}\rangle}} \tag{68}$$

is uniquely fixed by the following conditions:

$$g_e\,\frac{|z_{t(e),e}\rangle}{\sqrt{\langle z_{t(e),e}|z_{t(e),e}\rangle}}\,=\,\frac{|z_{s(e),e}]}{\sqrt{\langle z_{s(e),e}|z_{s(e),e}\rangle}},\quad g_e\,\frac{|z_{t(e),e}]}{\sqrt{\langle z_{t(e),e}|z_{t(e),e}\rangle}}\,=\,-\frac{|z_{s(e),e}\rangle}{\sqrt{\langle z_{s(e),e}|z_{s(e),e}\rangle}},\quad g_e\in\mathrm{SU}(2), \tag{69}$$

thus sending the source normalized spinor onto the dual of the target normalized spinor. Let us point out that if we impose the matching conditions $\langle z_{s(e),e}|z_{s(e),e}\rangle - \langle z_{t(e),e}|z_{t(e),e}\rangle = 0$ on the spinors, then the norm-factors can be dropped out of the previous equations. This truly means that the g_e's define the parallel transport of the spinors along the edges of the graph. This expression $g_e(z_{s(e),e}, z_{t(e),e})$ is clearly U(1)-invariant i.e invariant under the simultaneous multiplication by a phase of the two spinors:

$$z_{s(e),e} \to e^{i\theta_e}\, z_{s(e),e}, \qquad z_{t(e),e} \to e^{-i\theta_e}\, z_{t(e),e}.$$

Thus we can consider any function $\phi(g_e)$ as a function $\psi(z_{v,e})$. We would still need to check how the SU(2) gauge invariant of the $\phi(g_e)$ functionals are turned into the closure conditions for the wave-functions $\psi(z_{v,e})$.

We postpone a rigorous mathematical study of this issue and the resulting proof of the conjecture to future investigation [8]. Instead, here, we would like to focus on using this formula for the SU(2) group elements in terms of our spinors to express the holonomy operators of Loop Quantum Gravity in terms of the U(N) operators.

B. Reconstructing Holonomies

The group elements $g_e(z_{s(e),e}, z_{t(e),e}) \in \mathrm{SU}(2)$ that we constructed in the previous section are invariant under U(1) and thus commute with the matching conditions $E_e^{s(e)} - E_e^{t(e)}$ ensuring that the energy of the oscillators on the edge e at the vertex $s(e)$ is the same as at the vertex $t(e)$. However, they are obviously not invariant under SU(2) transformation. As well-known in loop (quantum) gravity, in order to construct SU(2)-invariant observables, we need to consider the trace of holonomies around closed loops, i.e the oriented product of group elements g_e along closed loops $\mathcal{L}$ on the graph:

$$G_{\mathcal{L}} \equiv \overrightarrow{\prod_{e\in\mathcal{L}}} g_e. \tag{70}$$

Let us assume for simplicity's sake that all the edges of the loop are oriented the same way, so that we can number the edges $e_1, e_2, ..e_n$ with $v_1 = t(e_n) = s(e_1)$, $v_2 = t(e_1) = s(e_2)$ and so on. Then, we can write explicitly the holonomy $G_{\mathcal{L}}$ in terms of the spinors:

$$\mathrm{Tr}\, G_{\mathcal{L}} = \mathrm{Tr}\, g(e_1)..g(e_n) = \mathrm{Tr}\, \frac{\prod_i (|z_{v_i,e_i}]\langle z_{v_{i+1},e_i}| - |z_{v_i,e_i}\rangle[z_{v_{i+1},e_i}|)}{\prod_i \sqrt{\langle z_{v_i,e_i}|z_{v_i,e_i}\rangle\langle z_{v_{i+1},e_i}|z_{v_{i+1},e_i}\rangle}}. \tag{71}$$

Now, instead of factorizing this expression per edge, let us group the terms per vertex:

$$\mathrm{Tr}\, G_{\mathcal{L}} = \sum_{r_i=0,1} (-1)^{\sum_i r_i} \frac{\prod_i \langle \varsigma^{r_{i-1}} z_{v_i,e_{i-1}} \,|\, \varsigma^{1-r_i} z_{v_i,e_i}\rangle}{\prod_i \sqrt{\langle z_{v_i,e_{i-1}}|z_{v_i,e_{i-1}}\rangle\langle z_{v_i,e_i}|z_{v_i,e_i}\rangle}}, \tag{72}$$

where the ς^{r_i} records whether we have the term $|z_{v_i,e_i}]\langle z_{v_{i+1},e_i}|$ or $|z_{v_i,e_i}\rangle[z_{v_{i+1},e_i}|$ on the edge e_i. Let us remember that ς is the (anti-unitary) map sending a spinor $|z\rangle$ to each dual $|z]$.

Now, depending of the specific values on the r_i parameters, the scalar products at the numerators are given by the matrix elements of M^i or Q^i at the vertex i:

$$\begin{array}{c|c|c} r_{i-1} & r_i & \langle \varsigma^{r_{i-1}} z_{v_i,e_{i-1}} \,|\, \varsigma^{1-r_i} z_{v_i,e_i}\rangle \\ \hline 0 & 0 & Q^i_{i,i-1} \\ 0 & 1 & M^i_{i-1,i} \\ 1 & 0 & M^i_{i,i-1} \\ 1 & 1 & \bar{Q}^i_{i,i-1} \end{array} \tag{73}$$

Since the matrices $M^i, Q_i, \bar{Q}^i$ are by definition SU(2)-invariant (they commute with the closure conditions), this provides a posteriori check that the holonomy $\mathrm{Tr}\, G_{\mathcal{L}}$ correctly provides a SU(2)-observables.

Taking into account the various possibilities for the signs $(-1)^{r_i}$, we can write the holonomy in a rather barbaric way:

$$\mathrm{Tr}\, G_{\mathcal{L}} = \sum_{r_i=0,1} (-1)^{\sum_i r_i} \frac{\prod_i r_{i-1} r_i \bar{Q}^i_{i,i-1} + (1-r_{i-1}) r_i M^i_{i-1,i} + r_{i-1}(1-r_i) M^i_{i,i-1} + (1-r_{i-1})(1-r_i) Q^i_{i,i-1}}{\prod_i \sqrt{\langle z_{v_i,e_i}|z_{v_{i+1},e_i}\rangle}}, \tag{74}$$

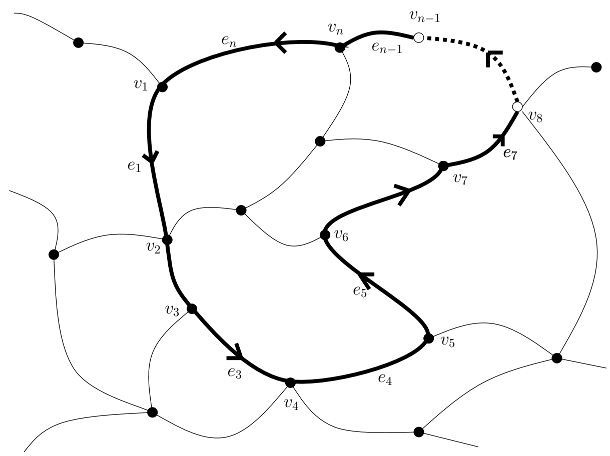

FIG. 2: The loop $\mathcal{L} = \{e_1, e_2, .., e_n\}$ on the graph Γ.

where actually only one of the four terms is selected for each set of $\{r_i\}$. To simplify the notations, we call $\mathcal{M}_{\mathcal{L}}^{\{r_i\}}$ each term for a fixed set $\{r_i\}$:

$$\begin{aligned}\mathcal{M}_{\mathcal{L}}^{\{r_i\}} &\equiv \prod_i r_{i-1} r_i \bar{Q}^i_{i,i-1} \,+\, (1-r_{i-1}) r_i M^i_{i-1,i} \,+\, r_{i-1}(1-r_i) M^i_{i,i-1} \,+\, (1-r_{i-1})(1-r_i) Q^i_{i,i-1} \\ &= \prod_i \langle \varsigma^{r_{i-1}} z_{v_i,e_{i-1}} \,|\, \varsigma^{1-r_i} z_{v_i,e_i} \rangle .\end{aligned} \tag{75}$$

Each of these quantities are still SU(2)-invariant observables and are also invariant under the $\mathrm{U}(1)^E$ transformations generated by the matching conditions. So there are genuine observables on the space of spin networks.

After having expressed the holonomy observable in terms of the spinors and $M, Q, \bar{Q}$ matrices at the classical level, our purpose is to promote it to a quantum operator and express the holonomy operator acting on spin network states in terms of the $\mathrm{U}(N)$-operators $E, F, F^\dagger$. In order to achieve this, looking at the vertex v and the pair of edges e, f, we simply have to quantize the matrix elements as:

$$\begin{aligned} M^v_{ef} &\rightarrow E^v_{ef}, \\ Q^v_{ef} &\rightarrow {F^v_{ef}}^\dagger, \\ \bar{Q}^v_{ef} &\rightarrow F^v_{ef}. \end{aligned} \tag{76}$$

Therefore the quantization of the holonomy observable is obvious apart from the factors at the denominator. First, we notice that the norm $\langle z_{v,e} | z_{v,e} \rangle$ for each edge e attached to v is simply the matrix element M^v_{ee} giving the total energy on the leg e for the intertwiner living at v. The natural quantization of these terms is thus E^v_{ee}. However, we need to take the inverse square-root of these operators and they do have a 0 eigenvalue. We must also face possible ordering ambiguities because all the E and F and $F^\dagger$ operators do not commute. In order to decide which ordering is right, we draw inspiration from the direct calculation of the holonomy operator for the 2-vertex graph done in [7] and conjecture the following expression:

Conjecture 2. *We can express the holonomy operator around a closed loop $\mathcal{L}$ (assuming that all the edges are oriented the same way) acting on spin network states as:*

$$\widehat{\mathrm{Tr}\, G_{\mathcal{L}}} \,=\, \sum_{r_i=0,1} (-1)^{\sum_i r_i} \mathcal{E}\, \widehat{\mathcal{M}}_{\mathcal{L}}^{\{r_i\}}\, \mathcal{E}, \tag{77}$$

with the operators

$$\mathcal{E} \,\equiv\, \frac{1}{\prod_i \sqrt{E_{e_i}+1}}$$

and

$$\widehat{\mathcal{M}}_{\mathcal{L}}^{\{r_i\}} \equiv \prod_i r_{i-1} r_i F^i_{i,i-1} \,+\, (1-r_{i-1}) r_i E^i_{i-1,i} \,+\, r_{i-1}(1-r_i) E^i_{i,i-1} \,+\, (1-r_{i-1})(1-r_i) {F^i_{i,i-1}}^\dagger .$$

First, we have written E_{e_i} without reference to any vertex. This is because spin network states satisfy the matching constraints on all edges $E^{s(e)}_{ee} = E^{t(e)}_{ee}$, therefore we write here $E_{e_i} \equiv E^{v_i}_{e_ie_i} = E^{v_i}_{e_{i-1}e_{i-1}}$. In particular, one can easily check that the operator $\prod_j \widehat{\mathcal{M}}_j$ commute with all the matching constraints $E^{v_i}_{e_ie_i} - E^{v_i}_{e_{i-1}e_{i-1}}$. Second, E_{e_i} is the energy operator of the oscillators living on the edge e_i, so it has a positive spectrum $\mathbb{N}$. Thus, the shifted operator $E_{e_i}+1$ is still Hermitian and has a strictly positive spectrum $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. Therefore, the operator $1/\sqrt{E_{e_i}+1}$ is well-defined.

Finally, we point out that the operator $\widehat{\mathrm{Tr}\, G_{\mathcal{L}}}$ defined as above is straightforwardly Hermitian.

In order to prove this conjecture, we could do a direct calculation of the action of the holonomy operator, check how it acts on all the intertwiners living at the vertices of the loops $\mathcal{L}$ and compare with the expression above. We believe that a more indirect check but certainly less painful and more enlightening would be to compute the algebra of our conjectured holonomy operators and compare it to the actual well-known holonomy algebra. We postpone this study to future investigation [8].

We nevertheless check our conjectured formula against the exact expression of the holonomy operators for the 2-vertex graph [7], and it seems that we have the exact same expressions apart from the sign factor $(-1)^{r_i}$. Let us look more carefully at this issue.

The 2-vertex graph consists in two vertices α and β, linked by N edges all oriented in the same direction from α to β. We number the edges $i = 1..N$. We now have $\mathrm{U}(N)$ operators acting at each vertex, $E^{(\alpha)}_{ij}, F^{(\alpha)}_{ij}, F^{(\alpha)\dagger}_{ij}$ and $E^{(\beta)}_{ij}, F^{(\beta)}_{ij}, F^{(\beta)\dagger}_{ij}$. Finally, the matching conditions to ensure that we are working with true spin network states are $E^{(\alpha)}_{ii} - E^{(\beta)}_{ii} = 0$ for all edges i.

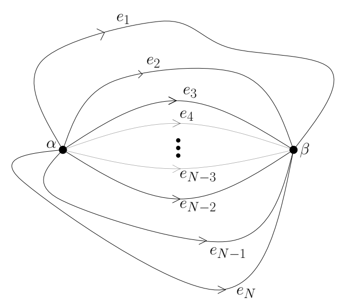

FIG. 3: The 2-vertex graph with vertices α and β and the N edges linking them.

Let us look at a basic loop consisting in two edges (ij). Then we apply our conjectured formula to get:

$$\widehat{\mathrm{Tr}\, G_{(ij)}} = -\frac{1}{\sqrt{E_i+1}\sqrt{E_j+1}} \left(F^{(\alpha)}_{ij}F^{(\beta)}_{ij} + E^{(\alpha)}_{ij}E^{(\beta)}_{ij} + E^{(\alpha)}_{ji}E^{(\beta)}_{ji} + F^{(\alpha)\dagger}_{ij}F^{(\beta)\dagger}_{ij}\right) \frac{1}{\sqrt{E_i+1}\sqrt{E_j+1}}. \tag{78}$$

This is the exact same expression as we have derived in the earlier work [7] apart from the global minus sign. This discrepancy is not an issue since it is only due to the difference of orientation. Indeed, our conjecture formula holds for all edges oriented the same way around the loop $\mathcal{L}$, while the formula derived in [7] assumes that the edges are all oriented from α to β. There is no problem with changing the orientations in our formula for the holonomy operator above: we multiply by a minus sign for each edge whose orientation we switch.

At the end of the day, the present framework is totally consistent with the full analysis of spin network states on the 2-vertex graph done in [7].

By expressing the holonomy operator around a closed loop in terms of the operators E_{ij}, F_{ij} and $F^\dagger_{ij}$ of the $\mathrm{U}(N)$ formalism, we have finally written a proper SU(2)-invariant operators acting on spin network states and not only on single intertwiner states as done up to now [1–4]. As we have said earlier, looking carefully at the expression of the holonomy operator, each term of the sum over $r_i = 0, 1$ is also SU(2)-invariant and commutes with the matching conditions. Moreover, we can forget about the factors $\mathcal{E}$ in the denominator, which comes from properly normalizing the spinors into order to define the group elements g_e. Finally, we are left with the operators $\widehat{\mathcal{M}}^{\{r_i\}}_{\mathcal{L}}$ for each set of values $\{r_i\}$, which we interpret as defining *generalized holonomy operators* in our $\mathrm{U}(N)$ formalism for loop quantum gravity. These operators are simply constructed as the product of E or F or $F^\dagger$ operators acting on the vertices

around the loop:

$$\widehat{\mathcal{M}}_{\mathcal{L}}^{\{r_i\}} \equiv \prod_i r_{i-1}r_iF^i_{i,i-1} + (1-r_{i-1})r_iE^i_{i-1,i} + r_{i-1}(1-r_i)E^i_{i,i-1} + (1-r_{i-1})(1-r_i){F^i_{i,i-1}}^\dagger. \tag{79}$$

These are the natural SU(2)-invariant operators acting on spin network states in the U(N) formalism. It is easy to see that they shift the spin j_e of the edges around the loop $e \in \mathcal{L}$ by $\pm\frac{1}{2}$. For instance, for $r_i = 0$ for all i around the loop, then:

$$\widehat{\mathcal{M}}_{\mathcal{L}}^{\{r_i=0\}} = \prod_i {F^i_{i,i-1}}^\dagger$$

raises all the spins around the loop by $+\frac{1}{2}$. On the other hand, for $r_i = 1$ for all i's, we decrease all the spins by $-\frac{1}{2}$:

$$\widehat{\mathcal{M}}_{\mathcal{L}}^{\{r_i=1\}} = \prod_i F^i_{i,i-1}.$$

Now, if we put mixed values around the loop $\mathcal{L}$, the corresponding operator $\widehat{\mathcal{M}}_{\mathcal{L}}^{\{r_i\}}$ increases the spins of the edges i with $r_i = 0$ and decreases the spins on the edges labeled by $r_i = 1$.

We can also reconstruct every $\widehat{\mathcal{M}}_{\mathcal{L}}^{\{r_i\}}$ operator from the holonomy operator $\widehat{\mathrm{Tr}\, G_{(ij)}}$ by suitable insertions of the energy operators E_{e_i} in order to select specific $\pm\frac{1}{2}$ shifts for the spins around the loop. This was done explicitly in the case of the 2-vertex graph in [7] and can get straightforwardly generalized to arbitrary graphs and loops.

Finally, the algebra of these generalized holonomy operators $\widehat{\mathcal{M}}_{\mathcal{L}}^{\{r_i\}}$ will be investigated elsewhere [8].

IV. CLASSICAL DYNAMICS FOR SPIN NETWORKS

A. A Classical Action for Spin Networks

Let us start by summarizing the classical setting for spin network states on a given graph Γ. Spin network states are V intertwiner states -one at each vertex v- glued together along the edges e so that they satisfy the matching conditions on each edge. The phase space consists with the spinors $z_{v,e}$ (where e are edges attached to the vertex v, i.e such that $v = s(e)$ or $v = t(e)$) which we constrain by the closure conditions $\vec{\mathcal{C}}^v$ at each vertex v and the matching conditions on each edge e. The corresponding action reads:

$$S_0^\Gamma[z_{v,e}] = \int dt \sum_v \sum_{e|v\in\partial e} \left(-i\langle z_{v,e}|\partial_t z_{v,e}\rangle + \langle z_{v,e}|\Lambda_v|z_{v,e}\rangle\right) + \sum_e \rho_e \left(\langle z_{s(e),e}|z_{s(e),e}\rangle - \langle z_{t(e),e}|z_{t(e),e}\rangle\right), \tag{80}$$

where the 2×2 Lagrange multipliers Λ_v satisfying $\mathrm{Tr}\,\Lambda_v = 0$ impose the closure constraints and the Lagrange multipliers $\rho_e \in \mathbb{R}$ impose the matching conditions. All the constraints are first class, they generate SU(2) transformations at each vertex and U(1) transformations on each edge e.

We can describe the same system parameterized by $N_v \times N_v$ unitary matrices U^v and the parameters λ_v. The matrix elements U^v_{ef} refer to pairs of edges e, f attached to the vertex v. The closure conditions are automatically encoded in the requirement that the matrices U^v are unitary. We still have to impose the matching conditions $M^{s(e)}_{ee} - M^{t(e)}_{ee} = 0$ on each edge e where the matrices $M^v = \lambda_v\, U^v \Delta U^{v\,-1}$ are functions of both λ_v and U^v. The action then reads:

$$S_0^\Gamma[\lambda_v, U^v] = \int dt \sum_v \left(-i\,\lambda_v \mathrm{Tr}\, U^v \Delta \partial_t U^{v\dagger} - \mathrm{Tr}\Theta_v\,(U^v U^{v\dagger} - \mathbb{I})\right) + \sum_e \rho_e (M^{s(e)}_{ee} - M^{t(e)}_{ee}), \tag{81}$$

where the ρ_e impose the matching conditions as before while the $N_v \times N_v$ matrices Θ_v are the Lagrange multipliers for the unitarity of the matrices U^v. Moreover, this action is invariant under the action of $\mathrm{SU}(2)\times \mathrm{U}(N_v - 2)$ at every vertex, which reduces the number of degrees of freedom of the matrices U^v to the spinors $z_{v,e}$ which are actually the two first columns of those matrices.

This free action describes the classical kinematics of spin networks on the graph Γ. Now, we would like to add interaction terms and a Hamiltonian to this action in order to define a non-trivial dynamics for the system. Such interaction terms need to be compatible with the closure conditions and the matching conditions i.e be invariant under SU(2) at each vertex v and U(1) on each edge. The natural candidates are the generalized holonomy observables

$\mathcal{M}_{\mathcal{L}}^{\{r_i\}}$ which we described in the previous section. Our proposal for a classical action for spin networks with non-trivial dynamics is thus:

$$S^{\Gamma}_{\gamma_{\mathcal{L}}^{\{r_i\}}} = S_0^{\Gamma} + \int dt \sum_{\mathcal{L},\{r_i\}} \gamma_{\mathcal{L}}^{\{r_i\}} \mathcal{M}_{\mathcal{L}}^{\{r_i\}}, \tag{82}$$

where the $\gamma_{\mathcal{L}}^{\{r_i\}}$ are the coupling constants giving the relative weight of each generalized holonomy in the full Hamiltonian. Let us point out that the generalized holonomies $\mathcal{M}_{\mathcal{L}}^{\{r_i\}}$ are a priori not independent from each other. We postpone the analysis of this issue to future investigation. Instead, we will study in more detail this classical action principle in the specific case of the 2-vertex graph.

B. A Matrix Model for the Dynamics on the 2-Vertex Graph

Coming back to the 2-vertex graph, we have the two vertices α, β linked with N edges. The corresponding classical phase space is parameterized by $2N$ spinors $z_i^{(\alpha)}$ and $z_i^{(\beta)}$. Then we need to impose the closure constraints on both vertices:

$$\sum_i |z_i^{(\alpha)}\rangle\langle z_i^{(\alpha)}| = \frac{1}{2}\sum \langle z_i^{(\alpha)}|z_i^{(\alpha)}\rangle\, \mathbb{I}, \qquad \sum_i |z_i^{(\beta)}\rangle\langle z_i^{(\beta)}| = \frac{1}{2}\sum \langle z_i^{(\beta)}|z_i^{(\beta)}\rangle\, \mathbb{I}, \tag{83}$$

and the matching conditions on all N edges:

$$\forall i, \quad \langle z_i^{(\alpha)}|z_i^{(\alpha)}\rangle \,=\, \langle z_i^{(\beta)}|z_i^{(\beta)}\rangle\,. \tag{84}$$

These constraints are not straightforward to solve explicitly, since the $z_i^{(\alpha)}, z_i^{(\beta)}$ are spinors and not just complex numbers. We can also write the corresponding action principle in terms of the unitary matrices U^α, U^β:

$$S_0[U^\alpha, U^\beta, \lambda_\alpha, \lambda_\beta] \equiv \int dt \left(-i\lambda_\alpha \mathrm{Tr} U^\alpha \Delta \partial_t U^{\alpha\dagger} - i\lambda_\beta \mathrm{Tr} U^\beta \Delta \partial_t U^{\beta\dagger} + \sum_i \rho_i \left(\lambda_\alpha (U^\alpha \Delta U^{\alpha\dagger})_{ii} - \lambda_\beta (U^\beta \Delta U^{\beta\dagger})_{ii}\right)\right), \tag{85}$$

where we have left implicit the constraints imposing the unitarity of U^α and U^β. It is clear that the matching conditions imply that $\lambda_\alpha = \lambda_\beta$. We can thus slightly simplify this action:

$$S_0[U^\alpha, U^\beta, \lambda] \,\equiv\, \int dt \left(-i\,\lambda \left[\mathrm{Tr}\, U^\alpha \Delta \partial_t U^{\alpha\dagger} \,+\, \mathrm{Tr}\, U^\beta \Delta \partial_t U^{\beta\dagger}\right] + \sum_i \rho_i \left[(U^\alpha \Delta U^{\alpha\dagger})_{ii} \,-\, (U^\beta \Delta U^{\beta\dagger})_{ii}\right]\right). \tag{86}$$

Geometrically, λ represents the total boundary area of the surface separating the two vertices, while U^α and U^β describe the shapes and deformations of the two intertwiners sitting at α and β.

We would like to add some dynamics on this basic setting of the 2-vertex graph. Elementary loops on this very simple graph are made of two edges (ij). Then given such a loop, we have four possibilities for our generalized holonomy observables $\mathcal{M}_{\mathcal{L}}^{\{r_i\}}$. The observable $Q^\alpha_{ij} Q^\beta_{ij}$ corresponds at the quantum level to raising the spins j_i and j_j on both edges i, j. Its conjugate $\bar{Q}^\alpha_{ij}\bar{Q}^\beta_{ij}$ will decrease the spins j_i and j_j at the quantum level. The observable $M^\alpha_{ij} M^\beta_{ij}$ will increase the spin j_i while decreasing the spin j_j. The final possibility is $M^\alpha_{ji} M^\beta_{ji} = \overline{M^\alpha_{ij} M^\beta_{ij}}$ simply reverses the role of the two edges i and j. Finally, our ansatz for a generic action with a non-trivial dynamics reads:

$$S[U^\alpha, U^\beta, \lambda] \,\equiv\, S_0[U^\alpha, U^\beta, \lambda] + \int dt \sum_{i,j} \left[\gamma^+_{ij}\, Q^\alpha_{ij} Q^\beta_{ij} + \gamma^-_{ij}\, \bar{Q}^\alpha_{ij}\bar{Q}^\beta_{ij} + \gamma^0_{ij}\, M^\alpha_{ij} M^\beta_{ij}, \right], \tag{87}$$

where the γ's are coupling constants and where we remind that the matrices M and Q are defined as $M = \lambda U \Delta U^\dagger$ and $Q = \lambda U \Delta_\epsilon {}^t U$. Further requiring that the new interaction terms defining the action's Hamiltonian be real, we need to impose further condition on the coupling constants:

$$\gamma^- = \overline{\gamma^+}, \qquad \gamma^0 = (\gamma^0)^\dagger.$$

In the previous work on spin networks on the 2-vertex graph [7], it was discussed to introduce an extra U(N) symmetry and further require that the dynamics of the system be invariant under that symmetry. This was then interpreted as imposing isotropy on the model. These coupled U(N) transformations act on the matrices at both vertices α and β:

$$\left| \begin{array}{l} U^\alpha \to VU^\alpha \\ U^\beta \to \bar{V}U^\beta \end{array} \right. \qquad \text{for} \quad V \in \mathrm{U}(N). \tag{88}$$

These U(N) transformations are generated by $\widehat{M}^\alpha_{ij} - \widehat{M}^\beta_{ji}$ and they reduce to the $\mathrm{U}(1)^N$ transformations generated by the matching constraints in the case that V is a unitary diagonal matrix.

The kinematical terms in $\mathrm{T}r\, U^\alpha \Delta \partial_t U^{\alpha\dagger}$ and $\mathrm{T}r\, U^\beta \Delta \partial_t U^{\beta\dagger}$ are obviously invariant under such transformations. The interaction terms also need to be U(N)-invariant. It is easy to check that this leaves only three possible U(N)-invariant terms made from all the generalized holonomy observables in the equation 87:

$$\gamma^+ \,\mathrm{T}r\, Q^\alpha Q^\beta + \gamma^- \,\mathrm{T}r\, \bar{Q}^\alpha \bar{Q}^\beta + \gamma^0 \,\mathrm{T}r\, M^{\alpha\,t} M^\beta \,, \tag{89}$$

where we remind that ${}^tM = \bar{M}$ and ${}^tQ = -Q$. The requirement to keep the Hamiltonian real imposes as above that $\gamma^- = \overline{\gamma^+}$ and $\gamma^0 \in \mathbb{R}$.

Finally, we need to deal with the matching conditions $M^\alpha_{ii} - M^\beta_{ii} = 0$. Imposing the invariance under the coupled U(N)-transformations implies imposing the full equality between the matrices M^α and ${}^tM^\beta$, and not only the equality of their matrix elements on the diagonal. This is a very strong condition, which relates the unitary matrices U^α and U^β to each other:

$$M^\alpha = {}^tM^\beta \;\Leftrightarrow\; U^\alpha \Delta U^{\alpha\dagger} = \overline{U^\beta} \Delta \overline{U^\beta}^\dagger \qquad \Rightarrow \quad U^\alpha \;=\; e^{i\phi}\, \overline{U^\beta}, \tag{90}$$

where ϕ is an arbitrary phase factor and the matrices U^α and U^β are defined up to $\mathrm{SU}(2)\times\mathrm{U}(N-2)$ transformations. This means that the spinors $z^{(\alpha)}$ and $z^{(\beta)}$ are also equal up to a global phase:

$$\bar{z}^{(\alpha)}_i \;=\; e^{i\phi}\, z^{(\beta)}_i, \tag{91}$$

which obviously solve the matching conditions. This phase $e^{i\phi}$ actually defines the SU(2) holonomy living on the edges between the two vertices.

Simply renaming the matrix $U^\beta \equiv U$, we can re-express the action for this U(N)-invariant sector in terms of λ, the phase ϕ and the matrix U. Actually it turns out that the unitary matrix U completely drops out and we are left with the two conjugated dynamical variables λ and ϕ:

$$S_{inv}[\lambda,\phi] \;=\; -2\int dt\, \left(\lambda\partial_t\phi - \lambda^2\left(\gamma^0 - \gamma^+ e^{2i\phi} - \gamma^- e^{-2i\phi}\right)\right), \tag{92}$$

with the Hamiltonian $H = \lambda^2(\gamma^0 - 2\gamma\cos(2\phi))$. This is an elementary action with simple equations of motion for the couple of variables (ϕ,λ). For the sake of simplicity, we will take $\gamma^+ = \gamma^- = \gamma \in \mathbb{R}$. Then we obtain the following equations of motion:

$$\begin{aligned} \partial_t\phi \;&=\; 2\lambda\,(\gamma^0 - 2\gamma\,\cos 2\phi)\,, \\ \partial_t\lambda \;&=\; -4\gamma\lambda^2\,\sin 2\phi\,. \end{aligned} \tag{93}$$

We easily identify two obvious classical solutions. First, $\lambda = 0$ (with ϕ constant and arbitrary) is the trivial solution. It has no evolution and corresponds to a vanishing total area. Second, we have the case where ϕ is constant, but λ does not vanish. In this case, we get:

$$\cos 2\phi \;=\; \frac{\gamma^0}{2\gamma}, \qquad \lambda = \frac{1}{(4\gamma\,\sin 2\phi)\,t + kk}\,,$$

where kk is a constant of integration. This solution exists iff $|\gamma^0| \leq 2\gamma$. Finally, we can try to solve the equations of motion more generally. We express λ in terms of $\partial_t\phi$ from the first equation, which we plug back into the second equation in order to finally obtain a differential equation on ϕ only:

$$\begin{aligned} \lambda \;&=\; \frac{\partial_t\phi}{2(\gamma^0 - 2\gamma\cos(2\phi))}\,, \\ (\gamma^0 - 2\gamma\cos(2\phi))\partial_t^2\phi \;&=\; 2\gamma\sin(2\phi)(\partial_t\phi)^2\,. \end{aligned}$$

Unfortunately, we haven't been able to solve this differential equation explicitly.

On the other hand, we would like to propose an alternative Hamiltonian, who leads to simpler equations of motion which we are able to solve exactly. Following what has been done in the quantum 2-vertex model presented in [7], we introduce a renormalized Hamiltonian:

$$\mathbf{h} \equiv \frac{1}{\lambda} H = \lambda(\gamma^0 - 2\gamma\cos(2\phi)). \tag{94}$$

This renormalized Hamiltonian is still SU(2) and U(N) invariant, and is related to the generalized holonomy observables through a factor $\frac{1}{\lambda}$. Using this new Hamiltonian $\mathbf{h}$, the equations of motion actually simplify to

$$\begin{aligned} \partial_t\phi &= \gamma^0 - 2\gamma\cos(2\phi), \\ \partial_t\lambda &= -4\gamma\lambda\sin(2\phi)\,. \end{aligned} \tag{95}$$

As in the quantum case [7], the properties of the renormalized Hamiltonian $\mathbf{h}$ are much more straightforward than the original Hamiltonian H. In fact, it is possible to solve exactly these differential equations. We solve for $\phi(t)$ analytically. Then, once we have the solution for ϕ, we can show that the following expression for λ in terms of ϕ solves the equations of motions:

$$\lambda = \frac{\epsilon}{\gamma^0 - 2\gamma\cos(2\phi)}\,, \tag{96}$$

where $\epsilon = \pm$ is a global sign. Let us point out that the equation of motion for λ only determines it up a global numerical factor. Then we should remember that λ is the total area and we always constrain it to be positive.

Now, we present the solutions for $\phi(t)$ (we have chosen the most convenient constants of integration due to the fact that this constants are just translations in the temporal variable), depending on the different values for the parameters γ^0 and γ:

Elliptic region ($|\gamma^0| > 2|\gamma|$) :
$$\phi(t) = -\arctan\left(\frac{(2\gamma-\gamma^0)\tan\left(t\sqrt{(\gamma^0)^2-4\gamma^2}\right)}{\sqrt{(\gamma^0)^2-4\gamma^2}}\right), \tag{97a}$$

Hyperbolic region ($|\gamma^0| < 2|\gamma|$) :
$$\phi(t) = -\arctan\left(\frac{\sqrt{4\gamma^2-(\gamma^0)^2}}{(2\gamma+\gamma^0)\tanh\left(t\sqrt{4\gamma^2-(\gamma^0)^2}\right)}\right), \tag{97b}$$

Parabolic region I ($\gamma^0 = 2\gamma$) :
$$\phi(t) = -\arctan\left(\frac{1}{4\gamma t}\right), \tag{97c}$$

Parabolic region II ($\gamma^0 = -2\gamma$) :
$$\phi(t) = -\arctan(4\gamma t)\,. \tag{97d}$$

Let us give a brief description of these solutions. First, to derive $\lambda(t)$ from those solutions for $\phi(t)$, we compute $\cos 2\phi = (1-\tan^2\phi)/(1+\tan^2\phi)$ and we plug it in the expression (96) above for λ in terms of $\cos 2\phi$. Then, the two cases I and II of the parabolic regime are very similar. In both cases, we get the same solution $\lambda(t)$ by taking $\epsilon = +$ in case I and $\epsilon = -$ in case II. Switching the sign ϵ in the two cases allows to keep a positive solution $\lambda(t) \geq 0$. We have named the different regions with the name of the conics because, indeed, the equation 96 is the equation for a conic with radial coordinate given by λ and polar coordinate 2ϕ, as we can appreciate in the figure 4. Then, the different values for the ratio $2\gamma/\gamma^0$ represent the different eccentricities corresponding to each of the conics. In the elliptic case, when $|\gamma^0| > 2|\gamma|$, we have a system in which the area λ has an oscillatory behavior. In the other two regimes (hyperbolic with $|\gamma^0| < 2|\gamma|$ and parabolic with $|\gamma^0| = 2|\gamma|$), the area shrinks under evolution, reaches a minimum value and then increases until infinity. As it was pointed out in [7], the quantum Hamiltonian of this 2-vertex model is mathematically (and physically) analogous to the gravitational part of the Hamiltonian in loop quantum cosmology (LQC). Following this analogy, we can interpret the results we obtained here as the classical model for the quantum big bounce found in LQC. Nevertheless, further investigation is needed in order to achieve a full understanding of the relation with the results derived in the LQC framework.

At this point, it would be enlightening to compare the phase space (ϕ, λ) and the dynamics defined by the action (92) above with the classical setting of loop cosmology (see e.g. [9]). In this sense, the framework presented here opens at least two interesting lines of research upon understanding the precise and explicit links between the 2-vertex framework and the loop cosmology.

First, we should go beyond the U(N)-invariant sector. Indeed, the action (87) defines the full classical kinematics and dynamics of spin network states on the 2-vertex graph. It is a non-trivial matrix model defined in terms of the

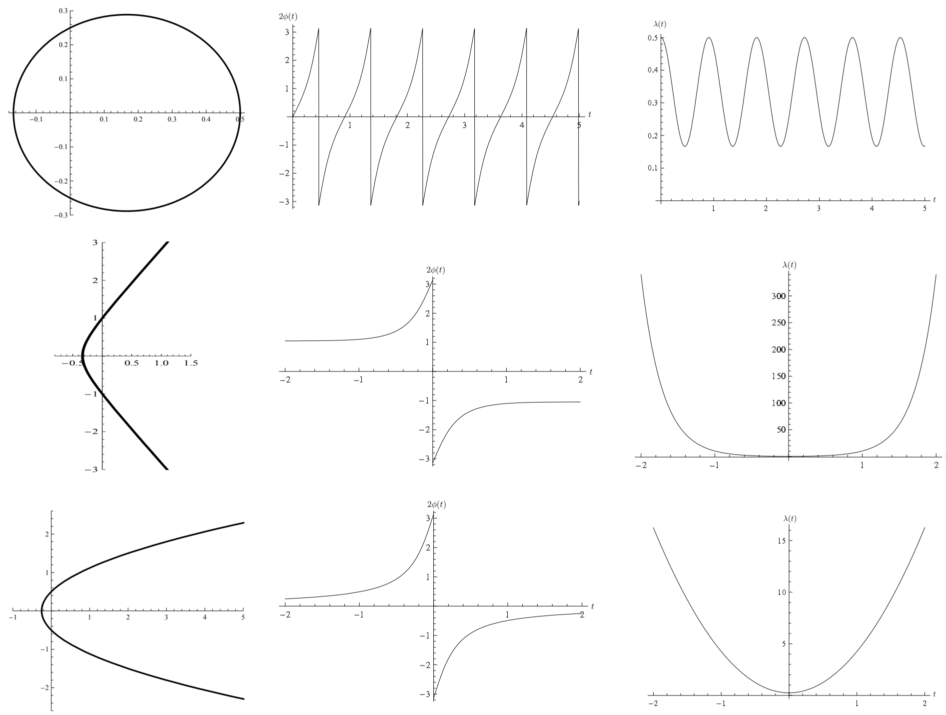

FIG. 4: We plot the behavior of $\phi(t)$ and $\lambda(t)$ (given by the equations 96 and 97) in the three different regimes for $\gamma = 1$ and respectively $\gamma^0 = 4$ (elliptic regime), $\gamma^0 = 1$ (hyperbolic regime) and finally $\gamma^0 = 2$ (parabolic regime). In the first column, we give the polar plots constructed by taking as polar coordinates $(2\phi, \lambda(\phi))$. The second column gives for $\phi(t)$ and the third one $\lambda(t)$. We observe in those plots the periodical behavior of λ (interpreted as the total area of the model) as a function of time in the elliptic case and a behavior analogous to a cosmological big bounce in the other two cases.

unitary matrices U^α and U^β and with quartic interaction terms. Moreover, even if we still choose a U(N)-invariant Hamiltonian of the the type $\gamma^+ \,\mathrm{Tr}\, Q^\alpha Q^\beta + \gamma^- \,\mathrm{Tr}\, \bar{Q}^\alpha \bar{Q}^\beta + \gamma^0 \,\mathrm{Tr}\, M^{\alpha\, t} M^\beta$, this will nevertheless induce non-trivial dynamics for the matrices U^α and U^β. It would be very interesting what kind of anisotropy does our model describe in the context of loop cosmology.

Second, our classical phase space expressed in terms of spinors or equivalently in terms of unitary matrices admit a straightforward quantization in terms of U(N) representations. This quantization scheme should be compared to the quantization procedure of loop quantum cosmology. This would help understanding the explicit relation between loop quantum cosmology and the full theory of loop quantum gravity.

Finally, a constant issue would be to couple matter degrees of freedom to our model, both in the specific case of the 2-vertex model and in the general case of spin network states on an arbitrary graph. The advantage of our approach here is that we can do it at the classical level in our spinor phase space before quantizing.

Conclusion

The U(N) framework introduced in [1–4] provides us with a new interesting way to describe the space of intertwiners for loop quantum gravity. Using this framework, it has been shown that it is possible to tackle some of the important issues in LQG, such as the construction of coherent states [3], dynamics [7] or the simplicity constraints for spinfoam models [4]. Our motivation for the present paper was to define the classical phase space (using spinors) underlying the U(N) framework and to introduce the corresponding classical action principle.

As it was suggested in [7], we explored in this paper the idea of considering the operators E_{ij} and F_{ij} coming from the quantization of matrix elements of a hermitian matrix M and an antisymmetric matrix Q that satisfy the same quadratic constraints as the operators themselves (up to quantum ordering terms). We gave the explicit expression for this matrices in terms of elements of a unitary matrix U and the parameter λ corresponding to the trace of the matrix M (whose quantum analog is the total area operator E). This allowed us to write these matrices in terms

of spinors defined from the matrix elements of the unitary matrix. This show how the so-called closure conditions for the spinors, introduced in [3], come from the unitarity requirement in the construction of our matrices M and Q. We then described the phase space in terms of the spinors, introduced the corresponding Poisson brackets and showed that the closure constraints generate the SU(2) action relevant to defining intertwiners states. We further computed the Poisson brackets of the matrix elements M_{ij} and Q_{ij} and compared them to the commutator algebra of the corresponding quantum operators E_{ij} and F_{ij}. Finally, we proposed an action from which we can derive the whole spinor phase space structure. We wrote it alternatively in terms of the spinors or in terms of the unitary matrix U and the classical boundary area λ. This way, we have reformulated the classical setting of a single SU(2) intertwiner as a unitary matrix model (as it was already suggested in the pioneer work [1]). Using this action principle, we finally showed that there is no non-trivial dynamics for a single intertwiner.

We moved on to the quantum level and showed how to perform the quantization of the classical spinor phase space in order to obtain the Hilbert space of intertwiners in terms of holomorphic (or alternatively anti-holomorphic) wave-functions. Having explored in detail the framework for a single intertwiner both at the classical and quantum level, we studied the gluing of those intertwiners and showed to define loop quantum gravity's spin-network states over an arbitrary graph as (anti-)holomorphic wave functions of spinors (appropriately constrained). We have postponed a more rigorous proof of the equivalence of the standard loop quantum gravity framework to our new $\mathrm{U}(N)$/spinor framework to future investigation [8]. Then, making use of the expression for SU(2) group elements in terms of spinors given in [6], we constructed the expression for loop gravity's holonomy observables in terms of spinors attached to each of the vertices of the graph and, finally, in terms of the elements of the matrices M and Q. We discuss the quantization of this expression and express the holonomy operator at the quantum level in terms of the E_{ij} and F_{ij} operators of the $\mathrm{U}(N)$ framework for intertwiners. We checked this formula against the formula derived for the (loop) quantum gravity 2-vertex model previously introduced and discussed by some of the authors [7].

Finally, we wrote an action for the classical setting of spin-network states on a general graph and we applied it to the special case of the simple graph with two vertices. Choosing a specific form for the interaction term, written in terms of the (generalized) holonomy observables, we obtained the classical description for the $\mathrm{U}(N)$ invariant Hamiltonian for the 2-vertex model considered in [7].

To summarize, we have proposed here a classical setting whose quantization in terms of (anti-)holomorphic functionals over constrained spinors describes intertwiners and spin network states. Furthermore, we proposed a classical action principle encoding the whole corresponding kinematical structure and possible dynamics for spin network states.

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[1] F. Girelli and E.R. Livine, *Reconstructing Quantum Geometry from Quantum Information: Spin Networks as Harmonic Oscillators*, Class.Quant.Grav. 22 (2005) 3295-3314 [arXiv:gr-qc/0501075]

[2] L. Freidel and E.R. Livine, *The Fine Structure of* SU(2) *Intertwiners from* $\mathrm{U}(N)$ *Representations*, Journ. Math. Phys. 51 (2010) 082502 [arXiv:0911.3553]

[3] L. Freidel and E.R. Livine, $\mathrm{U}(N)$ *coherent states for Loop Quantum Gravity*, arXiv:1005.2090

[4] M. Dupuis and E.R. Livine, *Revisiting the Simplicity Constraints and Coherent Intertwiners*, arXiv:1006.5666

[5] L. Freidel and S. Speziale, *Twisted geometries: A geometric parametrisation of SU(2) phase space*, arXiv:1001.2748

[6] L. Freidel and S. Speziale, *From twistors to twisted geometries*, arXiv:1006.0199

[7] E.F. Borja, J. Diaz-Polo, I. Garay and E.R. Livine *Dynamics for a 2-vertex Quantum Gravity Model*, arXiv:1006.2451

[8] L. Freidel, E.R. Livine and S. Speziale, *Generalized Holonomy Observables and Holomorphic/Anti-Holomorphic Splitting*, in preparation

[9] A. Ashtekar, *Loop Quantum Cosmology: An Overview*, Gen.Rel.Grav.41 (2009)707-741 [arXiv:0812.0177]